

TABLE OF CONTENTS

ALGEBRA: SEQUENCE AND SERIES	2
ALGEBRA: SEQUENCE AND SERIES SOLUTIONS.....	4
ALGEBRA: WORDED PROBLEMS	14
ALGEBRA: WORDED PROBLEMS SOLUTIONS.....	19
ALGEBRA: POLYNOMIALS, PARTIAL FRACTIONS AND INEQUALITIES.....	35
ALGEBRA: POLYNOMIALS, PARTIAL FRACTIONS AND INEQUALITIES SOLUTIONS.....	38
ALGEBRA: BINOMIAL EXPANSION	49
ALGEBRA: BINOMIAL EXPANSION SOLUTIONS.....	50
ALGEBRA: COMPLEX NUMBERS	54
ALGEBRA: COMPLEX NUMBERS SOLUTIONS	57
MATRICES AND DETERMINANTS.....	68
MATRICES AND DETERMINANTS SOLUTIONS.....	71
PLANE AND SPHERICAL TRIGONOMETRY	77
PLANE AND SPHERICAL TRIGONOMETRY SOLUTIONS	84
PLANE ANALYTIC GEOMETRY.....	105
PLANE ANALYTIC GEOMETRY SOLUTIONS.....	110
SOLID MENSURATION	123
SOLID MENSURATION SOLUTIONS	129
SPACE ANALYTIC GEOMETRY	150
SPACE ANALYTIC GEOMETRY SOLUTIONS	152
VECTORS	162
VECTORS SOLUTIONS.....	166
DIFFERENTIAL CALCULUS.....	176
DIFFERENTIAL CALCULUS SOLUTIONS.....	184
INTEGRAL CALCULUS	226
INTEGRAL CALCULUS SOLUTIONS.....	229
STATISTICS	247
STATISTICS SOLUTIONS	253
PROBABILITY	258
PROBABILITY SOLUTIONS	263

ALGEBRA: SEQUENCE AND SERIES

1. Find the n^{th} term of the arithmetic progression 6, 10, 14, ...

A. $2 + 4n$	C. $4 + 2n$
B. $2 - 4n$	D. $4 - 2n$
2. Find the sum up to the 10^{th} term of the arithmetic progression 6, 10, 14, ...

A. 250	C. 240
B. 120	D. 225
3. Find the n^{th} term of the arithmetic progression $\log 7, \log 14, \log 28, \dots$

A. $0.845 + 0.301n$	C. $-0.544 + 0.301n$
B. $0.544 + 0.301n$	D. $-0.544 - 0.301n$
4. Find the sum up to the 10^{th} term of the arithmetic progression $\log 7, \log 14, \log 28, \dots$

A. 18.442	C. 25.852
B. 26.397	D. 21.997

Situation: Logs are stacked so that there are 25 logs in the bottom row, 24 logs in the second row, and so on, decreasing by 1 log each row.

5. How many logs are stacked in the sixth row?

A. 21	C. 19
B. 20	D. 18
6. How many logs are there in all six rows?

A. 136	C. 137
B. 134	D. 135

Situation: The distance a ball rolls down a ramp each second is given by the arithmetic sequence whose n^{th} term $2n - 1$ in feet.

7. Find the distance the ball rolls during the 10^{th} second.

A. 18 ft.	C. 19 ft.
B. 20 ft.	D. 21 ft.
8. Find the total distance the ball travels in 10 seconds.

A. 120 ft.	C. 110 ft.
B. 100 ft.	D. 90 ft.

Situation: A contest offers 15 prizes. The 1st prize is P 5000, and each successive prize is P 250 less than the preceding prize.

9. What is the value of the 15^{th} prize?

A. 1250	C. 1500
B. 1750	D. 1625
10. What is the total amount of money distributed in prizes?

A. P 49,000	C. P 50,750
B. P 48,750	D. P 47,250

Situation: The 4^{th} and 7^{th} terms of an arithmetic sequence are 13 and 25.

11. Find the first term.

A. 1	C. 2
B. 3	D. 5
12. Find the common difference.

A. 1	C. 5
B. 4	D. 2
13. Find the 20^{th} term.

A. 81	C. 73
B. 77	D. 83

14. Insert 5 arithmetic means between -1 and 23.
 A. 3, 7, 11, 15, 19 C. -1, 3, 7, 11, 15
 B. 7, 10, 13, 16, 19 D. 1, 5, 9, 13, 17
15. An object dropped from a cliff will fall 16 feet the first second, 48 feet the second, 80 feet the third, and so on, increasing by 32 feet each second. What does the total distance the object will fall in 7 seconds?
 A. 874 ft. C. 748 ft.
 B. 847 ft. D. 784 ft.
16. A besieged fortress is held by 5700 men who have provisions for 66 days. If the garrison loses 20 men each day, how many days can the provision hold out?
 A. 70 days C. 67 days
 B. 76 days D. 80 days
17. In a racing contest, there are 240 cars with fuel provision for 15 hours each. Assuming a constantly hourly consumption for each car, how long will the fuel provision last if 8 cars withdraw from the race every hour?
 A. 72 hours C. 25 hours
 B. 23 hours D. 20 hours
18. How many numbers divisible by 4 lie between 70 and 203?
 A. 33 C. 34
 B. 35 D. 36
19. Find the sum of the numbers divisible by between 70 and 203.
 A. 4848 C. 8484
 B. 4488 D. 8844
20. Two men set out from a certain place going in the same direction. The first travels at a constant rate of 8 kilometers per hour, while the second goes 4 km for the first hour, 4.5 km the second hour, 5 km the third hour, and so on. After how many hours will the second man overtake the first?
 A. 18 hours C. 17 hours
 B. 20 hours D. 19 hours
21. Ten balls are placed in a straight line on the ground at intervals of 2 meters. Six meters from the end of the row a basket is placed. A boy starts from the basket and picks up the balls and carries them, one at a time to the basket. How far did he walk all in all?
 A. 120 m C. 250 m
 B. 130 m D. 300 m
22. Find the third term of a geometric sequence whose first term is 2 and whose fifth term is 162.
 A. 18 C. 9
 B. 6 D. 27
23. A basketball is dropped from a height of 10m. On each rebound it rises $\frac{2}{3}$ of the height from which it last fell. Determine the total distance travelled until it comes to rest.
 A. 45 m C. 50 m
 B. 60 m D. 75 m
24. Find the sum of the geometric progression 2, 6, 18, ... up to the 10th term.
 A. 10,682 C. 59,048
 B. 177,146 D. 6,560

ALGEBRA: SEQUENCE AND SERIES SOLUTIONS

1. Find the n^{th} term of the arithmetic progression 6, 10, 14, ...

First Solution:

Using the formula:

$$a_n = a_1 + (n - 1)d$$

$$a_1 = 6 \rightarrow \text{first term}$$

$$d = 4 \rightarrow \text{common difference}$$

We have:

$$a_n = 6 + (n - 1)(4)$$

$$a_n = 6 + 4n - 4$$

$$a_n = 2 + 4n$$

Second Solution:

Since the relationship between a_n and n is linear, hence we can use here the STAT Mode 3-2 $\rightarrow A + Bx$

For Linear Mathematical Model, we only need 2 points to define the function in the form of $y = A + Bx$.

Input:

x	y
1	6
2	10

Press AC.

Then press Shift 1 - 5 (Reg - Regression)

Then select 1: A and then select 2: B

$A = 2$; $B = 4$

Therefore, $y = A + Bx$

$$a_n = 2 + 4n$$

2. Find the sum up to the 10th term of the arithmetic progression 6, 10, 14, ...

First Solution:

By using the formula of the sum of an arithmetic progression of n terms:

$$s_n = \frac{[2a_1 + (n - 1)d]n}{2}$$

$$s_n = \frac{[2(6) + (10 - 1)(4)](10)}{2}$$

$$s_n = 240$$

Second Solution:

From the formula, it shows that the relationship between n and s_n is in quadratic form, so we can use STAT MODE 3-3 $\rightarrow A + Bx + Cx^2$

For Quadratic Mathematical Model, we need 3 points to define the function in the form of $y = A + Bx + Cx^2$.

Input:

x	y
1	6
2	$6 + 10 = 16$
3	$6 + 10 + 14 = 30$

For the sum of the first 10 terms:

Find the value of y which corresponds to the value of $x = 10$.

Hence $10\hat{y} = 240$.

3. Find the n^{th} term of the arithmetic progression $\log 7, \log 14, \log 28, \dots$

First Solution:

Simplifying:

$$\begin{aligned} a_n &= a_1 + (n - 1)d \\ a_n &= \log 7 + (n - 1)(\log 2) \\ a_n &= \log 7 + n \log 2 - \log 2 \\ a_n &= \log \left(\frac{7}{2}\right) + n \log 2 \\ a_n &= 0.544 + 0.301n \end{aligned}$$

Second solution:

Go to MODE 3-2:

Input:

x	y
1	$\log 7$
2	$\log 14$

Press AC.

Then press Shift 1 - 5 (Reg - Regression)

Then select 1: A and then select 2: B

$A = 0.544; B = 0.301$

Therefore, $y = A + Bx$

$$a_n = 0.544 + 0.301n$$

4. Find the sum up to the 10^{th} term of the arithmetic progression $\log 7, \log 14, \log 28, \dots$

First Solution:

By using the formula of the sum of an arithmetic progression of n terms:

$$\begin{aligned} S_n &= \frac{[2a_1 + (n - 1)d]n}{2} \\ S_n &= \frac{[2 \log 7 + (10 - 1)(\log 2)](10)}{2} \\ S_n &= 21.997 \end{aligned}$$

Second Solution:

From the formula, it shows that the relationship between n and s_n is in quadratic form, so we can use STAT MODE 3-3 $\rightarrow A + Bx + Cx^2$

For Quadratic Mathematical Model, we need 3 points to define the function in the form of $y = A + Bx + Cx^2$.

Input:

x	y
1	$\log 7$
2	$\log 7 + \log 14$
3	$\log 7 + \log 14 + \log 28$

For the sum of the first 10 terms:

Find the value of y which corresponds to the value of $x = 10$.

Hence $10\hat{y} = 21.997$.

Situation: Logs are stacked so that there are 25 logs in the bottom row, 24 logs in the second row, and so on, decreasing by 1 log each row.

5. How many logs are stacked in the sixth row?

First Solution:

The relationship between the order of rows and number of logs follows an arithmetic progression.

Let a_1 be the number of logs in the bottom row, that is 25; $a_2 = 24$; and $a_3 = 23$, so that the arithmetic sequence is 25, 24, 23, ...

Therefore, the number of logs in the sixth row is:

$$\begin{aligned} a_1 &= 25, d = -1 \\ a_n &= a_1 + (n - 1)d \\ a_6 &= 25 + (6 - 1)(-1) \\ a_6 &= 20 \end{aligned}$$

Second Solution:

Go to MODE 3-2.

Input to the x-column the order of rows of logs and to the y-column the number of logs in that particular row, that is:

x	y
1	25
2	24

Press AC.

To solve for the number of logs in the sixth row, press $6\hat{y} = 20$.

6. How many logs are there in all six rows?

First Solution:

By using the formula of the sum of an arithmetic progression of n terms:

$$\begin{aligned} S_n &= \frac{[2a_1 + (n - 1)d]n}{2} \\ S_n &= \frac{[2(25) + (6 - 1)(-1)](6)}{2} \\ S_n &= 135 \end{aligned}$$

Second Solution:

From the formula, it shows that the relationship between n and s_n is in quadratic form, so we can use STAT MODE 3-3 $\rightarrow A + Bx + Cx^2$

For Quadratic Mathematical Model, we need 3 points to define the function in the form of $y = A + Bx + Cx^2$.

Input:

x	y
1	25
2	$25 + 24 = 49$
3	$25 + 24 + 23 = 72$

For the sum of the first 6 terms:

Find the value of y which corresponds to the value of $x = 6$.

Hence $6\hat{y} = 135$.

Situation: The distance a ball rolls down a ramp each second is given by the arithmetic sequence whose n^{th} term $2n - 1$ in feet.

7. Find the distance the ball rolls during the 10th second.

First Solution:

Let: a_n be the distance traveled by ball in a certain interval at any n^{th} second

$n = n^{\text{th}}$ second

The distance traveled in the first second is $2n - 1$; hence $a_1 = 2(1) - 1 = 1$ foot; $a_2 = 2(2) - 1 = 3$ feet; $d = 3 - 1 = 2$ feet.

Therefore, using the formula:

$$a_n = a_1 + (n - 1)d$$

We have

$$a_{10} = 1 + (10 - 1)(2)$$

$$a_{10} = 19 \text{ feet}$$

Second Solution:

Go to MODE 3-2:

Input:

x	y
1	1
2	3

Press AC.

Then press $10\hat{y} = 19$ feet.

8. Find the total distance the ball travels in 10 seconds.

First Solution:

By using the formula of the sum of an arithmetic progression of n terms:

$$s_n = \frac{[2a_1 + (n - 1)d]n}{2}$$

$$s_n = \frac{[2(1) + (10 - 1)(2)](10)}{2}$$

$$s_n = 100 \text{ feet}$$

Second Solution:

From the formula, it shows that the relationship between n and s_n is in quadratic form, so we can use STAT MODE 3-3 $\rightarrow A + Bx + Cx^2$

For Quadratic Mathematical Model, we need 3 points to define the function in the form of $y = A + Bx + Cx^2$.

Input:

x	y
1	1
2	$1 + 3 = 4$
3	$1 + 3 + 5 = 9$

For the sum of the first 10 terms:

Find the value of y which corresponds to the value of $x = 10$.

Hence $10\hat{y} = 100$ feet.

Situation: A contest offers 15 prizes. The 1st prize is P 5000, and each successive prize is P 250 less than the preceding prize.

9. What is the value of the 15th prize?

First Solution:

Using the formula:

$$a_n = a_1 + (n - 1)d$$

$$a_{15} = 5000 + (15 - 1)(-250)$$

$$a_{15} = P 1,500$$

Second Solution:

Go to MODE 3-2:

Input:

x	y
1	5000
2	4750

Then press $15\hat{y} = P 1,500$.

10. What is the total amount of money distributed in prizes?

First Solution:

By using the formula of the sum of an arithmetic progression of n terms:

$$s_n = \frac{[2a_1 + (n - 1)d]n}{2}$$

$$s_n = \frac{[2(5000) + (15 - 1)(-250)](15)}{2}$$

$$s_n = P 48,750$$

Second Solution:

From the formula, it shows that the relationship between n and s_n is in quadratic form, so we can use STAT MODE 3-3 $\rightarrow A + Bx + Cx^2$

For Quadratic Mathematical Model, we need 3 points to define the function in the form of $y = A + Bx + Cx^2$.

Input:

x	y
1	5000
2	$5000 + 4750 = 9750$
3	$5000 + 4750 + 4500 = 14250$

For the sum of the first 15 terms:

Find the value of y which corresponds to the value of x = 15.

Hence $15\hat{y} = P 48,750$.

Situation: The 4th and 7th terms of an arithmetic sequence are 13 and 25.

11. Find the first term.

12. Find the common difference.

13. Find the 20th term.

First Solution:

Using the formula:

$$a_n = a_1 + (n - 1)d$$

$$a_4 = 13; n = 4$$

$$13 = a_1 + (4 - 1)d$$

$$13 = a + 3d \rightarrow \text{equation 1}$$

$$\begin{aligned}
 a_7 &= 25; n = 7 \\
 25 &= a_1 + (7 - 1)d \\
 25 &= a_1 + 6d \rightarrow \text{equation 2} \\
 \begin{cases} a_1 + 3d = 13 \\ a_1 + 6d = 25 \end{cases}
 \end{aligned}$$

Solve simultaneously using MODE 5-1:

$$\begin{array}{c}
 \left| \begin{array}{cc|c} 1 & 3 & 13 \\ 1 & 6 & 25 \end{array} \right. \\
 a_1 = 1 \rightarrow \text{first term} \\
 d = 4 \rightarrow \text{common difference}
 \end{array}$$

Hence, the formula results to:

$$\begin{aligned}
 a_n &= 1 + (n - 1)(4) \\
 a_{20} &= 1 + (20 - 1)(4) \\
 a_{20} &= 77
 \end{aligned}$$

Second Solution:

Go to MODE 3-2:

Input:

x	y
4	13
7	25

For the first term, press $1\hat{y} = 1$

For the common difference, press Shift - 1 - 5 - 2 $\rightarrow B = 4$

For the 20th term, press $20\hat{y} = 77$

14. Insert 5 arithmetic means between -1 and 23.

Solution:

Let $a_1 = -1$ and $a_7 = 23$

$$\begin{aligned}
 a_n &= a_1 + (n - 1)d \\
 23 &= -1 + (7 - 1)d \\
 d &= 4 \rightarrow \text{common difference}
 \end{aligned}$$

Therefore, the 5 arithmetic means of -1 and 23 are 3, 7, 11, 15, and 19.

15. An object dropped from a cliff will fall 16 feet the first second, 48 feet the second, 80 feet the third, and so on, increasing by 32 feet each second. What does the total distance the object will fall in 7 seconds?

First Solution:

Using the formula:

$$\begin{aligned}
 s_n &= \frac{[2a_1 + (n - 1)d]n}{2}; a_1 = 16; d = 32 \\
 s_n &= \frac{[2(16) + (7 - 1)(32)](7)}{2} \\
 s_n &= 784 \text{ ft}
 \end{aligned}$$

Second Solution:

Go to MODE 3-3:

Input:

x	y
1	16
2	$16 + 48 = 64$
3	$16 + 48 + 80 = 144$

Then press $7\hat{y} = 784 \text{ ft.}$

16. A besieged fortress is held by 5700 men who have provisions for 66 days. If the garrison loses 20 men each day, how many days can the provision hold out?

Solution:

To complete the provision for besieged fortress, it requires
(5700 men)(66 days) = 376,200 man – days of work.

First day: (5700 men)(1 day) = 5700 man – days

Second day: (5680 men)(1 day) = 5680 man – days

Third day: (5660 men)(1 day) = 5660 man – days

And so on.

So, from this, the sum of all man-days should be equal to 376,200 man-days.

$$5700 + 5680 + 5660 + \dots = 376,200$$

Using the formula:

$$s_n = \frac{[2a_1 + (n - 1)d]n}{2}$$

$$376,200 = \frac{[2(5700) + (n - 1)(-20)](n)}{2}$$

$$n = 76 \text{ days}$$

17. In a racing contest, there are 240 cars with fuel provision for 15 hours each. Assuming a constantly hourly consumption for each car, how long will the fuel provision last if 8 cars withdraw from the race every hour?

Solution:

The total number of hours that cars can travel is equal to
(240 cars)(15 hours/car) = 3,600 hours.

First hour: (240 cars)(1 hour/car) = 240 hours.

Second hour: (232 cars)(1 hour/car) = 232 hours.

Third hour: (224 cars)(1 hour/car) = 224 hours.

And so on.

So, from this, the sum of all the time (in hours) is equal to 3600 hours.

$$240 + 232 + 224 + \dots = 3600$$

Using the formula:

$$s_n = \frac{[2a_1 + (n - 1)d]n}{2}$$

$$3,600 = \frac{[2(2400) + (n - 1)(-8)](n)}{2}$$

$$n = 25 \text{ hours}$$

18. How many numbers divisible by 4 lie between 70 and 203?

First solution:

The first number that is divisible by 4 after 70 is 72.

The nearest number that is divisible by 4 before 203 is 200.

Using the formula:

$$a_n = a_1 + (n - 1)d$$

$$200 = 72 + (n - 1)(4)$$

$$n = 33 \text{ terms}$$

Second solution:

The first term is $a_1 = 72$; since $d = 4$, we can say that $a_2 = 72 + 4 = 76$.

Using MODE 3-2:

Input:

x	y
1	72
2	76

Then press $200\hat{y} = 33$.

19. Find the sum of the numbers divisible by between 70 and 203.

First Solution:

Using the formula:

$$S_n = \frac{[2a_1 + (n - 1)d]n}{2}$$

$$S_n = \frac{[2(72) + (33 - 1)(4)](33)}{2}$$

$$n = 4,488$$

Second Solution:

Go to MODE 3-3:

Input:

x	y
1	72
2	72 + 76 = 148
3	72 + 76 + 80 = 228

Then press $33\hat{y} = 4,488$.

20. Two men set out from a certain place going in the same direction. The first travels at a constant rate of 8 kilometers per hour, while the second goes 4 km for the first hour, 4.5 km the second hour, 5 km the third hour, and so on. After how many hours will the second man overtake the first?

Solution:

For the second man to overtake the first man, the distance traveled by second man should be equal to the distance traveled by first man.

Let:

D = distance traveled by both men at any given time t

t = time of travel

For the first man traveling at a constant rate:

$$D = 8t$$

For the second man traveling at an increasing distance per succeeding hour:

$$D = \frac{[2a_1 + (n - 1)d]n}{2}; a_1 = 4, d = 0.5$$

$$D = \frac{[2(4) + (t - 1)(0.5)]t}{2}$$

Equating the two equations:

$$8t = \frac{[2(4) + (t - 1)(0.5)]t}{2}$$

$$t = 17 \text{ hours}$$

21. Ten balls are placed in a straight line on the ground at intervals of 2 meters. Six meters from the end of the row a basket is placed. A boy starts from the basket and picks up the balls and carries them, one at a time to the basket. How far did he walk all in all?

Solution:

The distance of the first ball to the basket is 6 m, then of the second ball, that is 6 m + 2 m = 8 m, and then of the third ball is 8 m + 2 m = 10 m, and so on.

From this, we can see that the terms follow an arithmetic progression, in which the first term, $a_1 = 6$, and the common difference $d = 2$, so that $n = 10$ for the 10th ball.

Using the formula:

$$s_n = \frac{[2a_1 + (n - 1)d]n}{2}$$

$$s_n = \frac{[2(6) + (10 - 1)2](10)}{2}$$

$$s_n = 150 \text{ m}$$

Or by using MODE 3-3:

Input:

x	y
1	6
2	6 + 8 = 14
3	6 + 8 + 10 = 24

Then press $10\hat{y} = 150 \text{ m}$.

But since the boy walks back and forth, therefore the total distance he traveled is 300 m.

22. Find the third term of a geometric sequence whose first term is 2 and whose fifth term is 162.

First Solution:

Using the formula for the n^{th} term of a geometric progression:

$$a_n = a_1 r^{n-1}$$

$$a_1 = 2; a_5 = 162$$

$$162 = 2r^{5-1}$$

$$r = 3$$

Therefore, for the third term:

$$a_3 = 2(3)^{3-1}$$

$$a_3 = 18$$

Second Solution:

Geometric progression follows the mathematical model of $y = AB^x$:

Go to MODE 3-6:

Input:

x	y
1	2
5	162

Then press $3\hat{y} = 18$.

23. A basketball is dropped from a height of 10m. On each rebound it rises $\frac{2}{3}$ of the height from which it last fell. Determine the total distance travelled until it comes to rest.

Solution:

Distance traveled on the first fall: 10 m

Distance traveled on the second fall: $10 \left(\frac{2}{3}\right) = \frac{20}{3} \text{ m}$

Thus, it follows an infinite geometric progression ($r < 1$).

To get the sum of the terms of an infinite geometric progression, the formula is given as:

$$s_n = \frac{a_1}{1 - r}; r \neq 1$$

But since the progression starts when $a_1 = \frac{20}{3}$, that's because the ball bounces, to compensate the distance traveled both upward and downward, the total distance traveled is expressed as:

$$D = 10 + 2 \left(\frac{20/3}{1 - 2/3} \right)$$

$$D = 50 \text{ m}$$

24. Find the sum of the geometric progression 2, 6, 18, ... up to the 10th term.

First Solution:

From the formula of the sum of a geometric progression of n terms:

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

$$a_1 = 2; r = 3$$

$$S_{10} = \frac{2(1 - 3^{10})}{1 - 3}$$

$$S_{10} = 59,048$$

Second Solution:

Go to MODE 3-6:

Input:

x	y
1	2
2	6

Then press Shift – 1 – 5 – 1 → A (Sto A)

Then press Shift – 1 – 5 – 2 → B (Sto B)

Then go back to MODE 1 and then press:

$$\sum_{x=1}^{10} (AB^x) = 59,048$$

ALGEBRA: WORDED PROBLEMS

- The difference between two numbers is 24 and their sum is 60. Find the smaller number.
A. 20
B. 24
C. 18
D. 27
- Find the smaller number, when two consecutive odd numbers such that thrice the smaller number exceeds the larger by 12.
A. 10
B. 5
C. 7
D. 9
- A certain two-digit number is 1 less than five times the sum of its digit. If 9 were added to the number, its digits would be reversed. Find the number.
A. 34
B. 29
C. 43
D. 47
- The sum of three numbers is 51. If the first number is divided by the second, the quotient is 2 and the remainder 5; but if the second number is divided by the third, the quotient is 3 and the remainder 2. Find the largest number.
A. 14
B. 33
C. 20
D. 37
- The sum of the digits of a 3-digit number is 12, the hundreds digit is twice the unit digit. If 198 is subtracted from the number, the order of the digits will be reversed. Find the hundreds digit.
A. 4
B. 6
C. 2
D. 0
- A boy is one third as old as his brother and 8 years younger than his sister. The sum of their ages is 38 years. How old is the boy?
A. 18 years old
B. 14 years old
C. 6 years old
D. 11 years old
- Letty is 10 years older than Cory who is half as old as Ben. If the total of their ages is 54 years, find the age of Ben.
A. 21 years old
B. 22 years old
C. 11 years old
D. 12 years old
- In a film, the actor Charles Coburn plays an elderly "uncle" character criticized for marrying a woman when he is 3 times her age. He wittily replies "Ah, but in 20 years time I shall only be twice her age" How old is the "uncle"?
A. 62 years old
B. 58 years old
C. 60 years old
D. 64 years old
- A man estimates that it will take him 7 days to roof his house. A professional roofer estimates that it will take him 4 days to roof the same house. How long will it take if they work together?
A. 2.55 days
B. 3.57 days
C. 2.10 days
D. 4.12 days

10. If 1000 articles of a given type can be turned out by first machine in 8 hours, by a second in 5 hours, and by a third in 4 hours, how long will it take to turn out the articles with all machines working?
- A. 1.937 hours C. 1.379 hours
 B. 1.739 hours D. 1.793 hours
11. A new machine that deposits cement for a road requires 12 hours to complete a one-half mile section of road. An older machine requires 16 hours to pave the same amount of road. After depositing cement for 4 hours, the new machine develops a mechanical problem and quits working. The older machine is brought into place and continues the job. How long does it take the older machine to complete the job?
- A. 11.667 hours C. 10.667 hours
 B. 12.333 hours D. 10.333 hours
12. John drove to a distant city in 5 hours. When he returned, there was less traffic, and the trip took only 3 hours. If John averaged 26 kph faster on the return trip, how fast did he drive each way?
- A. 42.125 kph C. 37.325 kph
 B. 40.875 kph D. 39 kph
13. Suzi drove home at 60 kph, but her brother Jim, who left at the same time, could drive at only 48 kph. When Suzi arrived, Jim still had 60 kilometers to go. How far did Suzi drive?
- A. 250.667 km C. 300 km
 B. 320 km D. 225 km
14. Two cars leave Pima Community College traveling in opposite directions. One car travels at 60 kph and the other at 64 kph. In how many hours will they be 310 kilometers apart?
- A. 3 hours C. 3.5 hours
 B. 2.5 hours D. 1.75 hours
15. One morning, John drove 5 hours before stopping to eat. After lunch, he increased his speed by 10 kph. If he completed a 430-km trip in 8 hours of driving time, how fast did he drive in the morning?
- A. 35 kph C. 65 kph
 B. 30 kph D. 50 kph
16. A motorboat goes 5 km upstream in the same time it requires to go 7 km downstream. If the river flows at 2 kph, find the speed of the boat in still water.
- A. 10 kph C. 12 kph
 B. 13.33 kph D. 60 kph
17. A plane can fly 340 kph in still air. If it can fly 200 kilometers downwind in the same amount of time it can fly 140 kilometers upwind, find the velocity of the wind.
- A. 10 kph C. 12 kph
 B. 13.33 kph D. 60 kph
18. A plane leaves an airport traveling at an average speed of 240 kilometers per hour. How long will it take a second plane traveling the same route at an

- average speed of 600 kilometers per hour to catch up with the first plane if it leaves 3 hours later?
- A. 2 hours C. 3 hours
B. 1 hour D. 5 hours
19. Marlene rides her bicycle to her friend Jon's house and returns home by the same route. Marlene rides her bike at constant speeds of 6 kph on level ground, 4 kph when going uphill and 12 kph when going downhill. If her total time riding was 1 hour, how far is it to Jon's house?
- A. 5 km C. 7 km
B. 8 km D. 3 km
20. An executive flew in the corporate jet to a meeting in a city 1500 kilometers away. After traveling the same amount of time on the return flight, the pilot mentioned that they still had 300 kilometers to go. The air speed of the plane was 600 kilometers per hour. How fast was the wind blowing? (Assume that the wind direction was parallel to the flight path and constant all day.)
- A. 50 kph C. 75 kph
B. 40.125 kph D. 66.667 kph
21. A car radiator has a 6-liter capacity. If the liquid in the radiator is 40% antifreeze, how much liquid must be replaced with pure antifreeze to bring the mixture up to a 50% solution?
- A. 4.0 L C. 2.7 L
B. 3.1 L D. 1.0 L
22. An automobile engine can run on a mixture of gasoline and a substitute fuel. If gasoline costs P 3.50 per gallon and the substitute fuel costs P 2 per gallon, what percent of the mixture must be substitute fuel to bring the cost down to P 2.75 per gallon?
- A. 10% C. 50%
B. 25% D. 66.67%
23. How many liters of water must evaporate to turn 12 liters of a 24% salt solution into a 36% solution?
- A. 4 L C. 2.7 L
B. 3.1 L D. 1.0 L
24. A forester mixes gasoline and oil to make 2 gallons of mixture for his two-cycle chainsaw engine. This mixture is 32 parts gasoline and 1-part two-cycle oil. How much gasoline must be added to bring the mixture to 40 parts gasoline and 1-part oil?
- A. 0.5 gal C. 1.939 gal
B. 0.061 gal D. 1.059 gal
25. How many ounces of pure gold that costs P850 per ounce must be mixed with 25 ounces of a gold alloy that costs P 500 per ounce to make a new alloy that costs P 725 per ounce?
- A. 45 ounces C. 30 ounces
B. 50.125 ounces D. 39 ounces
26. How many grams of pure silver must a silversmith mix with a 45% silver alloy to produce 200 grams of a 50% silver alloy?
- A. 181.818 grams C. 118.292 grams

- B. 18.181 grams D. 81.818 gram
27. At how many minutes after 3:00 p.m. will the minute hand overtake the hour hand?
 A. 3:16:21.82 pm C. 3:32:43.38 pm
 B. 3:08:10.55 pm D. 3:04:05.27 pm
28. At what time between 4 and 5 o'clock are the hands of a clock opposite each other?
 A. 4:45:27.16 C. 4:31:49.05
 B. 4:54:32.73 D. 4:40:54.32
29. At what time between 4 and 5 o'clock are the hands of a clock coincident?
 A. 4:27:16.36 C. 4:21:49.09
 B. 4:12:16.36 D. 4:08:10.91
30. At what time between 4 and 5 o'clock are the hands of a clock at right angles for the first time?
 A. 4:05:27.27 C. 4:38:10.91
 B. 4:43:38.18 D. 4:49:05.45
31. It is between 3 and 4 o'clock, and in 20 minutes the minute hand will be as much after the hour hand as it is now behind it. What is the time?
 A. 3:13:38.18 C. 3:09:05.45
 B. 3:06:21.82 D. 3:14:32.73
32. A woman invests P 37,000, part at 8% and the rest at 9.5 % annual interest. If the 9.5% investment provides P 452.50 more income than the 8% investment, how much is invested at the 8% rate?
 A. P 19500 C. P 20250
 B. P 17500 D. P 22500
33. Machine to mill a brass plate has a setup cost of P 600 and a unit cost of P3 for each plate manufactured. A bigger machine has a setup cost of P800 but a unit cost of only P2 for each plate manufactured. Find the break point.
 A. 200 C. 250
 B. 300 D. 600
34. A man has three sums of money invested, one at 12%, one at 10%, the last at 8%. His total annual income from three investments is P 2,100. The first investment yields as much as the other two combined. If he could receive 1% more on each investment his annual income would be increased by P 202.50. How much is his investment at the 12 % interest?
 A. P 5000 C. P 6500
 B. P 8750 D. P 6875
35. An investment of P 4600 is made at an annual simple interest rate of 6.8%. How much additional money must be invested at an annual simple interest rate of 9% so that the total interest earned is 8% of the total investment?
 A. P 4475 C. P 6625
 B. P 5000 D. P 5520
36. A and B submitted separate proposals for the construction of a bridge, A offering a lower price for the winning bid. Had A and B reduce their bid prices by 5% and 10% respectively, A would still have won the bid, but the

difference in their bids would have been reduced by 3000 pesos. If the sum total of the bids is 90000 pesos, what is the bid of A?

- A. P 50000
- B. P 62500
- C. P 40000
- D. P 27500

37. Two pipes are used to fill a water storage tank. The first pipe can fill the tank in 4 hours, and the two pipes together can fill the tank in 2 hours less time than the second pipe alone. How long would it take for the second pipe to fill the tank?

- A. 4 hours
- B. 2.5 hours
- C. 2 hours
- D. 5 hours

38. Scott and Laura have both invested some money. Scott invested P3,000 more than Laura and at a 2% higher interest rate. If Scott received P800 annual interest and Laura received P400, how much did Scott invest?

- A. P 3000
- B. P 8000
- C. P 2500
- D. P 10500

39. One of the important cities of the ancient world was Babylon. Greek historians wrote that the city was square-shaped. Its area numerically exceeded its perimeter by about 124. Find its dimensions in miles.

- A. 9.31 miles
- B. 6.94 miles
- C. 8.94 miles
- D. 13.31 miles

40. Small fishing boat heads to a point 24 km downriver and then returns. The current moves at the rate of 3 kilometers per hour. If the trip up and back takes 6 hours and the boat keeps a constant speed relative to the water, what is the speed of the boat?

- a. 8 kph
- B. 12 kph
- C. 9 kph
- D. 4 kph

ALGEBRA: WORDED PROBLEMS SOLUTIONS

Number Problems

1. The difference between two numbers is 24 and their sum is 60. Find the smaller number.

Solution:

Let x = first number

Let y = second number

$$\begin{cases} x - y = 24 \\ x + y = 60 \end{cases}$$

$$x = 42$$

$$y = 18$$

Therefore, the two numbers are **18** and **42**.

2. Find the smaller number, when two consecutive odd numbers such that thrice the smaller number exceeds the larger by 12.

Solution:

Let:

x = smaller odd number

$x + 2$ = larger odd number

From the statement, it follows that:

$$3x = (x + 2) + 12$$

Solving for the value of x :

$$x = 7$$

$$x + 2 = 9$$

Therefore, the answer is **7**.

3. A certain two-digit number is 1 less than five times the sum of its digit. If 9 were added to the number, its digits would be reversed. Find the number.

Solution:

Let x = tens digit; y = units digit

represent a two-digit number in terms of its digit:

$$\text{Number} = 10x + y$$

From the statement,

$$10x + y = 5(x + y) - 1$$

$$10x + y + 9 = 10y + x \rightarrow \text{equation 1}$$

Simplifying equation 1 we have,

$$5x - 4y = -1 \rightarrow \text{equation 1}$$

$$9x - 9y = -9 \rightarrow \text{equation 2}$$

Simplifying equation 2 we have,

$$x - y = -1 \rightarrow \text{equation 2}$$

Solving equations 1 and 2 simultaneously:

$$\begin{cases} 5x - 4y = -1 \\ x - y = -1 \end{cases}$$

$$x = 3; y = 4$$

Therefore, the number is **34**.

4. The sum of three numbers is 51. If the first number is divided by the second, the quotient is 2 and the remainder 5; but if the second number is divided by the third, the quotient is 3 and the remainder 2. Find the largest number.

Solution:

Let x = first number

Let y = second number

Let z = third number

$$x + y + z = 51 \rightarrow \text{equation 1}$$

$$\frac{x}{y} = 2 + \frac{5}{y} \rightarrow \text{equation 2}$$

$$\frac{y}{z} = 3 + \frac{2}{z} \rightarrow \text{equation 3}$$

Simplifying equation 2:

$$\left[\frac{x}{y} - \frac{5}{y} = 2 \right] (y)$$

$$x - 5 = 2y$$

$$x - 2y = 5 \rightarrow \text{equation 2}$$

Simplifying equation 3:

$$\left[\frac{y}{z} = 3 + \frac{2}{z} \right] (z)$$

$$y = 3z + 2$$

$$y - 3z = 2 \rightarrow \text{equation 3}$$

Solving simultaneously:

$$\begin{cases} x + y + z = 51 \\ x - 2y = 5 \\ y - 3z = 2 \end{cases}$$

$$\mathbf{x = 33}$$

$$\mathbf{y = 14}$$

$$\mathbf{z = 4}$$

The largest number is **33**.

5. The sum of the digits of a 3-digit number is 12, the hundreds digit is twice the unit digit. If 198 is subtracted from the number, the order of the digits will be reversed. Find the hundreds digit.

Solution:

Let x = hundreds digits

Let y = tens digit

Let z = units digit

$$x + y + z = 12$$

$$x = 2z$$

$$100x + 10y + z - 198 = 100z + 10y + x$$

$$\begin{cases} x + y + z = 12 \\ x - 2z = 0 \\ 99x - 99z = 198 \end{cases}$$

Solving simultaneously:

$$\mathbf{x = 4}$$

$$\mathbf{y = 6}$$

$$\mathbf{z = 2}$$

Thus, the number is 462. The hundreds digit is **4**.

Age Problems

6. A boy is one third as old as his brother and 8 years younger than his sister. The sum of their ages is 38 years. How old is the boy?

Solution:

Let x = boy's age

Let y = brother's age

Let z = sister's age

$$x = \frac{y}{3}$$

$$x = z - 8$$

$$x + y + z = 38$$

$$\begin{cases} 3x - y = 0 \\ x - z = -8 \\ x + y + z = 38 \end{cases}$$

$$x = \mathbf{6 \text{ years old}}$$

$$y = 18 \text{ years old}$$

$$z = 14 \text{ years old}$$

7. Letty is 10 years older than Cory who is half as old as Ben. If the total of their ages is 54 years, find the age of Ben.

Solution:

Let x = Letty's age

Let y = Cory's age

Let z = Ben's age

$$x = y + 10$$

$$y = \frac{z}{2}$$

$$x + y + z = 54$$

$$\begin{cases} x - y = 10 \\ 2y - z = 0 \\ x + y + z = 54 \end{cases}$$

$$x = 21 \rightarrow \text{Letty}$$

$$y = 11 \rightarrow \text{Cory}$$

$$z = \mathbf{22} \rightarrow \mathbf{Ben}$$

8. In a film, the actor Charles Coburn plays an elderly "uncle" character criticized for marrying a woman when he is 3 times her age. He wittily replies "Ah, but in 20 years time I shall only be twice her age" How old is the "uncle"?

Solution:

Let x = uncle's age; y = woman's age

	Present age	Age after 20 years
Uncle	x	$x+20$
Woman	y	$y+20$

$$x = 3y$$

$$(x + 20) = 2(y + 20)$$

$$\begin{cases} x - 3y = 0 \\ x - 2y = 20 \end{cases}$$

$$x = \mathbf{60 \text{ years old}}$$

$$y = 20 \text{ years old}$$

Work Problems

9. A man estimates that it will take him 7 days to roof his house. A professional roofer estimates that it will take him 4 days to roof the same house. How long will it take if they work together?

Solution:

$$\text{Rate of man} = 1/7$$

$$\text{Rate of Professional Roofer} = 1/4$$

Let: t = time to complete the whole job

$$\text{Combined rate} = \frac{1}{7} + \frac{1}{4}$$

Therefore,

$$\begin{aligned} \left(\frac{1}{7} + \frac{1}{4}\right)t &= 1 \\ \frac{11}{28}t &= 1 \\ \mathbf{t} &= \mathbf{2.55 \text{ days}} \end{aligned}$$

10. If 1000 articles of a given type can be turned out by first machine in 8 hours, by a second in 5 hours, and by a third in 4 hours, how long will it take to turn out the articles with all machines working?

Solution:

$$\text{Rate of first machine} = \frac{1}{8}$$

$$\text{Rate of second machine} = \frac{1}{5}$$

$$\text{Rate of third machine} = \frac{1}{4}$$

t = time to complete the 1000 articles

$$\begin{aligned} \left(\frac{1}{8} + \frac{1}{5} + \frac{1}{4}\right)t &= 1 \\ \mathbf{t} &= \mathbf{1.739 \text{ hours}} \end{aligned}$$

11. A new machine that deposits cement for a road requires 12 hours to complete a one-half mile section of road. An older machine requires 16 hours to pave the same amount of road. After depositing cement for 4 hours, the new machine develops a mechanical problem and quits working. The older machine is brought into place and continues the job. How long does it take the older machine to complete the job?

Solution:

$$\text{Rate of new machine} = \frac{1}{12}$$

$$\text{Rate of old machine} = \frac{1}{16}$$

t = number of hours the older machine to complete the job

$$\begin{aligned} \frac{1}{12}(4) + \frac{1}{16}(t) &= 1 \\ \mathbf{t} &= \mathbf{10.667 \text{ hours}} \end{aligned}$$

Motion Problems

12. John drove to a distant city in 5 hours. When he returned, there was less traffic, and the trip took only 3 hours. If John averaged 26 kph faster on the return trip, how fast did he drive each way?

Solution:

Let:

x = velocity of John in driving to a distant city in kph

$x + 26$ = velocity of John when he returns in kph

Using the basic equation:

$$\text{distance} = \text{velocity} \times \text{time}$$

$$d = 5x \rightarrow \text{equation 1}$$

$$d = (x + 26)(3) \rightarrow \text{equation 2}$$

Equating:

$$5x = (x + 26)(3)$$

$$\mathbf{x = 39 \text{ kph}}$$

13. Suzi drove home at 60 kph, but her brother Jim, who left at the same time, could drive at only 48 kph. When Suzi arrived, Jim still had 60 kilometers to go. How far did Suzi drive?

Solution:

Velocity of Suzi = 60 kph

Velocity of Jim = 48 kph

t = time of travel of Suzi to go home

$$d_s = 60t \rightarrow \text{Suzi}$$

$$d_j = 48t \rightarrow \text{Jim}$$

Since Jim is 60 km behind Suzi, therefore:

$$d_s = d_j + 60$$

$$60t = 48t + 60$$

$$t = 5 \text{ hours}$$

Therefore, the distance traveled by Suzi is:

$$d_s = 60(5) = \mathbf{300 \text{ km}}$$

14. Two cars leave Pima Community College traveling in opposite directions. One car travels at 60 kph and the other at 64 kph. In how many hours will they be 310 kilometers apart?

Solution:

Speed of car 1 = 60 kph

Speed of car 2 = 64 kph

Distance traveled by car 1 = $60t$

Distance traveled by car 2 = $64t$

The total distance traveled by the two cars is 310 km:

$$60t + 64t = 310$$

$$\mathbf{t = 2.5 \text{ hours}}$$

15. One morning, John drove 5 hours before stopping to eat. After lunch, he increased his speed by 10 kph. If he completed a 430-km trip in 8 hours of driving time, how fast did he drive in the morning?

Solution:

Let:

x = speed of John in the morning

$x + 10$ = speed of John after lunch

distance covered by John in the morning: $d_1 = 5x$

distance covered by John after lunch: $d_2 = (x + 10)(3)$

Adding two distances,

$$5x + 3(x + 10) = 430$$

$$x = \mathbf{50 \text{ kph}}$$

16. A motorboat goes 5 km upstream in the same time it requires to go 7 km downstream. If the river flows at 2 kph, find the speed of the boat in still water.

Solution:

Distance traveled upstream = 5 km

Distance traveled downstream = 7 km

Speed of river's current = 2 kph

Let: V_B = speed of boat in still water

Hence,

$$\text{speed of boat going upstream} = V_B - 2$$

$$\text{speed of boat going downstream} = V_B + 2$$

Let: t = time of travel

Therefore,

$$(V_B - 2)(t) = 5$$

$$(V_B + 2)(t) = 7$$

Dividing the equation, we have,

$$\frac{(V_B - 2)(t)}{(V_B + 2)(t)} = \frac{5}{7}$$

$$\frac{V_B - 2}{V_B + 2} = \frac{5}{7}$$

$$V_B = \mathbf{12 \text{ kph}}$$

17. A plane can fly 340 kph in still air. If it can fly 200 kilometers downwind in the same amount of time it can fly 140 kilometers upwind, find the velocity of the wind.

Solution:

Speed of plane in still air = 340 kph

Distance traveled in downwind = 200 km

Distance traveled in upwind = 140 km

Let: V_w = speed of wind

Relative speed of plane in downwind = $(340 + V_w)$

Relative speed of plane in upwind = $(340 - V_w)$

Let: t = time of travel

Distance traveled in downwind,

$$(340 + V_w)t = 200$$

Distance traveled in upwind,

$$(340 - V_w)t = 140$$

Dividing the equation to eliminate t:

$$\frac{(340 + V_w)t}{(340 - V_w)t} = \frac{200}{140}$$

$$\frac{(340 + V_w)}{(340 - V_w)} = \frac{200}{140}$$

$$\mathbf{V_w = 60 \text{ kph}}$$

18. A plane leaves an airport traveling at an average speed of 240 kilometers per hour. How long will it take a second plane traveling the same route at an average speed of 600 kilometers per hour to catch up with the first plane if it leaves 3 hours later?

Solution:

Speed of the first plane = 240 km/hr

Speed of the second plane = 600 km/hr

Let: t = time for the second plane to catch up the first plane

For the second plane to catch up the first plane, the distance traveled by the first plane must be equal to the distance traveled by the second plane.

$$\text{distance traveled by first plane} = 240(t + 3)$$

$$\text{distance traveled by second plane} = 600t$$

Equating,

$$240(t + 3) = 600t$$

$$\mathbf{t = 2 \text{ hours after the second plane leaves}}$$

19. Marlene rides her bicycle to her friend Jon's house and returns home by the same route. Marlene rides her bike at constant speeds of 6 kph on level ground, 4 kph when going uphill and 12 kph when going downhill. If her total time riding was 1 hour, how far is it to Jon's house?

Solution:

Speed on level ground = 6 kph

Speed on uphill = 4 kph

Speed on downhill = 12 kph

Let:

s_1 = distance traveled by bicycle on level ground in going to Jon's house

s_2 = distance traveled by bicycle on uphill in going to Jon's house

s_3 = distance traveled by bicycle on downhill in going to Jon's house

By equation:

$$s = vt$$

$$t = \frac{s}{v}$$

$$\frac{s_1}{6} = \text{time traveled on level ground in going to Jon's house}$$

$$\frac{s_2}{4} = \text{time traveled uphill in going to Jon's house}$$

$$\frac{s_3}{12} = \text{time traveled downhill going to Jon's house}$$

Hence, the total travel time in going to Jon's house is:

$$\frac{s_1}{6} + \frac{s_2}{4} + \frac{s_3}{12}$$

When returning home:

$$\frac{s_1}{6} = \text{time traveled on level ground returning home}$$

$$\frac{s_2}{12} = \text{time traveled downhill returning home}$$

$$\frac{s_3}{4} = \text{time traveled uphill returning home}$$

Hence, the total travel time returning home is:

$$\frac{s_1}{6} + \frac{s_2}{12} + \frac{s_3}{4}$$

Note that the speed for uphill and downhill will be interchanged, because all uphill will become downhill if you return in that same route, vice versa, and on level ground on course same speed.

Hence,

$$\left(\frac{s_1}{6} + \frac{s_2}{4} + \frac{s_3}{12}\right) + \left(\frac{s_1}{6} + \frac{s_2}{12} + \frac{s_3}{4}\right) = 1 \text{ hour}$$

$$\frac{s_1}{3} + \frac{s_2}{3} + \frac{s_3}{3} = 1$$

$$s_1 + s_2 + s_3 = \mathbf{3 \text{ km}}$$

20. An executive flew in the corporate jet to a meeting in a city 1500 kilometers away. After traveling the same amount of time on the return flight, the pilot mentioned that they still had 300 kilometers to go. The air speed of the plane was 600 kilometers per hour. How fast was the wind blowing? (Assume that the wind direction was parallel to the flight path and constant all day.)

Solution:

Total distance between two places = 1500 km

Speed of plane in air = 600 km/hr

Let: V_w = speed of the wind

Since the return flight the plane was not able to complete the 1500 km distance, on the same amount of time to go on that place, we can conclude that the plane travels slower on return flight. Hence we can say that the plane travels against the wind.

Total speed of plane against the wind = $(600 - V_w)$

Total speed of the plane along the wind = $(600 + V_w)$

Let: t = time of travel

$$\begin{cases} (600 + V_w)t = 1500 \\ (600 - V_w)t = 1200 \end{cases}$$

Dividing the equation to eliminate t :

$$\frac{(600 + V_w)}{(600 - V_w)} = \frac{1500}{1200}$$

$$V_w = \mathbf{66.67 \text{ kph}}$$

Mixture Problems

21. A car radiator has a 6-liter capacity. If the liquid in the radiator is 40% antifreeze, how much liquid must be replaced with pure antifreeze to bring the mixture up to a 50% solution?

Solution:

Let: x = amount of pure antifreeze to be substituted

Amount of antifreeze in the original solution = $6(0.4) = 2.4$ L

The amount of antifreeze that will remain in the solution if we draw a volume equal to amount of pure antifreeze, before replacing a pure antifreeze is:

$$2.4 - 0.4x$$

And putting the pure antifreeze in the solution, hence the amount of antifreeze in the solution will become

$$2.4 - 0.4x + x(1) = 0.5(6)$$

$$\mathbf{x = 1 \text{ liter}}$$

22. An automobile engine can run on a mixture of gasoline and a substitute fuel. If gasoline costs P 3.50 per gallon and the substitute fuel costs P 2 per gallon, what percent of the mixture must be substitute fuel to bring the cost down to P 2.75 per gallon?

Solution:

Let:

x = volume of gasoline in every 1 gallon of solution

y = volume of substitute fuel in every 1 gallon of solution

Hence,

$$\begin{cases} x + y = 1 \rightarrow \text{eq. by volume} \\ 3.5x + 2y = 2.75 \rightarrow \text{eq. by total cost} \end{cases}$$

Solving simultaneously,

$$x = 0.5 \text{ gal}$$

$$y = 0.5 \text{ gal}$$

Therefore,

$$\frac{0.5 \text{ gal}}{1 \text{ gal}} = 50\% \text{ of gasoline}$$

$$\frac{\mathbf{0.5 \text{ gal}}}{\mathbf{1 \text{ gal}}} = \mathbf{50\% \text{ of substitute fuel}}$$

23. How many liters of water must evaporate to turn 12 liters of a 24% salt solution into a 36% solution?

Solution:

Let: x = amount of water to be evaporated

The amount of salt after evaporation will remain the same hence,

$$0.36(12 - x) = 2.88$$

$$\mathbf{x = 4 \text{ liters}}$$

24. A forester mixes gasoline and oil to make 2 gallons of mixture for his two-cycle chainsaw engine. This mixture is 32 parts gasoline and 1-part two-cycle oil. How much gasoline must be added to bring the mixture to 40 parts gasoline and 1-part oil?

Solution:

Let:

x = volume of gasoline in original solution

y = volume of oil in original solution

$$\begin{cases} x + y = 2 \\ \frac{x}{y} = \frac{32}{1} \end{cases}$$

Hence the system of equation will become

$$\begin{cases} x + y = 2 \\ x - 32y = 0 \end{cases}$$

$x = 1.939$ gal of gasoline

$y = 0.061$ gal of oil

Let:

z = amount of gasoline to be added

$$\frac{1.939 + z}{0.061} = \frac{40}{1}$$

$z = 0.50$ gal

25. How many ounces of pure gold that costs P850 per ounce must be mixed with 25 ounces of a gold alloy that costs P 500 per ounce to make a new alloy that costs P 725 per ounce?

Solution:

Let x = weight of pure gold to be mixed with 25 ounces of gold, in ounce
 cost of pure gold in the new alloy = $850x$

cost of 25 ounces of gold alloy in the new alloy = $25(500) = 12500$

Hence,

$$850x + 12500 = 725(x + 25)$$

$x = 45$ ounces

26. How many grams of pure silver must a silversmith mix with a 45% silver alloy to produce 200 grams of a 50% silver alloy?

Solution:

x = mass of pure silver

y = mass of 45% silver alloy

$x + y = 200$ → equation due to total mass of alloy

$x + 0.45y = 0.5(200)$ → equation due to total amount of silver

Hence, solving simultaneously,

$x = 18.18$ grams → pure silver

$y = 181.818$ grams → 45% silver

Clock Problems

27. At how many minutes after 3:00 p.m. will the minute hand overtake the hour hand?

Solution:

Let: x = time in minutes after 3:00 pm

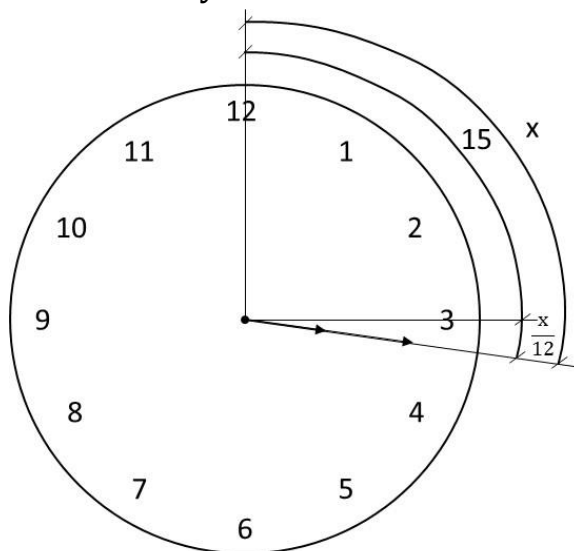
In one hour, the minute hand rotates 360 degrees, or:

$$\text{rate of minute hand} = 360^\circ/\text{hour} = 6^\circ/\text{minute}$$

On the other hand, the hour hand rotates 30 degrees in one hour, or:

$$\text{rate of hour hand} = 30^\circ/\text{hour} = 0.5^\circ/\text{minute}$$

If we relate the rate of the two hands, we can say that in one unit the minute hand travels, the hour hand only travels one-twelfth of a unit.



Referring to the figure:

$$15 + \frac{x}{12} = x$$

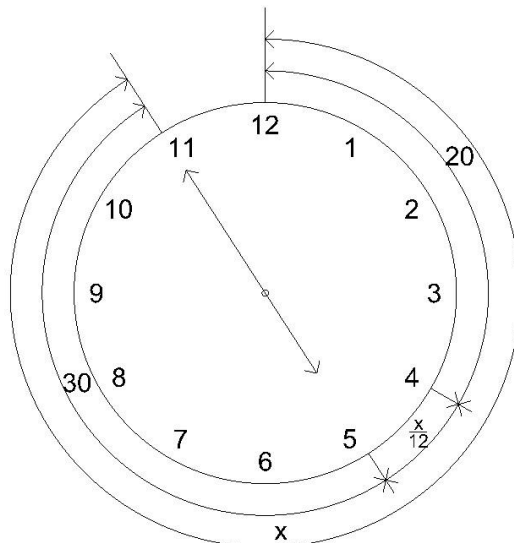
$$\mathbf{x = 16.36}$$

Therefore, the answer is **3:16:21.82 pm**.

28. At what time between 4 and 5 o'clock are the hands of a clock opposite each other?

Solution:

Let: x = time after 4 o'clock



Referring to the figure:

$$20 + \frac{x}{12} + 30 = x$$

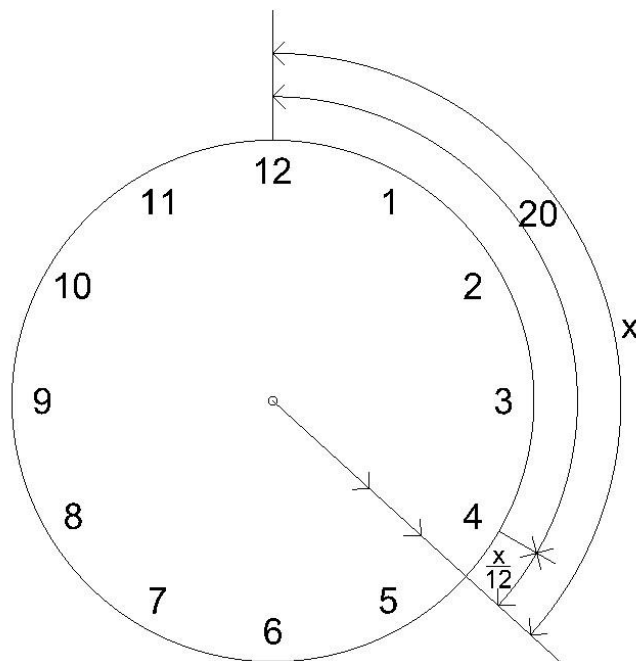
$$\mathbf{x = 54.55}$$

Therefore, the time is **4:54:32**.

29. At what time between 4 and 5 o'clock are the hands of a clock coincident?

Solution:

Let x = time after 4 o'clock



Referring to the figure:

$$20 + \frac{x}{12} = x$$

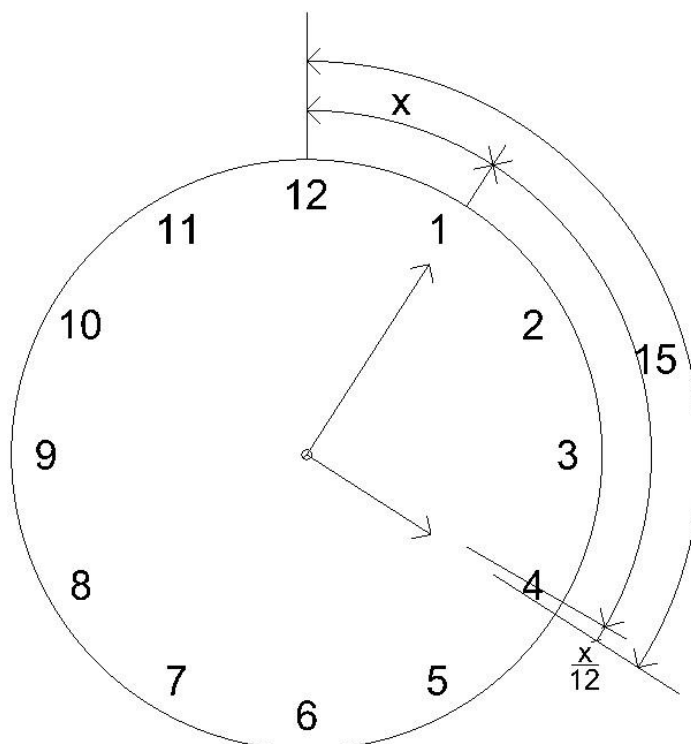
$$x = 21.82 \text{ min}$$

Therefore, the time is **4:21:49.09**.

30. At what time between 4 and 5 o'clock are the hands of a clock at right angles for the first time?

Solution:

Let: x = time after 4 o'clock



Referring to the figure:

$$20 + \frac{x}{12} = x + 15$$

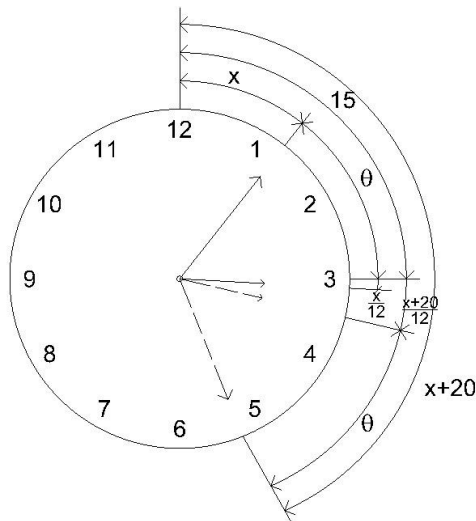
$$x = 5.45$$

Therefore, the time is **4:05:27.27**.

31. It is between 3 and 4 o'clock, and in 20 minutes the minute hand will be as much after the hour hand as it is now behind it. What is the time?

Solution:

Let x = time in minutes after 3 o'clock



Referring to the figure:

Time now (solid line):

$$\theta = 15 + \frac{x}{12} - x$$

Time in 20 minutes (broken line):

$$\theta = (x + 20) - 15 - \frac{(x + 20)}{12}$$

Equating the two equations will yield to:

$$15 + \frac{x}{12} - x = (x + 20) - 15 - \frac{(x + 20)}{12}$$

$$\mathbf{x = 6.36}$$

Therefore, the time is **3:06:21.82**.

Investment Problems

32. A woman invests P 37,000, part at 8% and the rest at 9.5 % annual interest. If the 9.5% investment provides P 452.50 more income than the 8% investment, how much is invested at the 8% rate?

Solution:

Let:

x = amount interested at 8%

y = amount interested at 9.5%

$$x + y = 37000$$

$$0.095y = 0.08x + 452.50$$

$$\mathbf{x = 17500}$$

$$\mathbf{y = 19500}$$

33. Machine to mill a brass plate has a setup cost of P 600 and a unit cost of P3 for each plate manufactured. A bigger machine has a setup cost of P800 but a unit cost of only P2 for each plate manufactured. Find the break point.

Solution:

Let x = number of brass plate

For break point, the total manufacturing cost for smaller machine must be equal to the total manufacturing cost for bigger machine.

Total manufacturing cost for smaller machine = $600 + 3x$

Total manufacturing cost for bigger machine = $800 + 2x$

$$600 + 3x = 800 + 2x$$

$$\mathbf{x = 200}$$

34. A man has three sums of money invested, one at 12%, one at 10%, the last at 8%. His total annual income from three investments is P 2,100. The first investment yields as much as the other two combined. If he could receive 1% more on each investment his annual income would be increased by P 202.50. How much is his investment at the 12 % interest?

Solution:

Let:

x = amount invested at 12%

y = amount invested at 10%

z = amount invested at 8%

Total annual income from three investments = 2100

$$0.12x + 0.10y + 0.08z = 2100 \rightarrow \text{equation 1}$$

First investment yields as much as the other two:

$$0.12x = 0.10y + 0.08z \rightarrow \text{equation 2}$$

Receiving 1% more on each investment, his annual income would be increased by P 202.50:

$$0.13x + 0.11y + 0.09z = 2302.50 \rightarrow \text{equation 3}$$

Solving the three equations, simultaneously:

$$\begin{cases} 0.12x + 0.10y + 0.08z = 2100 \\ 0.12x - 0.10y - 0.08z = 0 \\ 0.13x + 0.11y + 0.09z = 2302.50 \end{cases}$$

Solving simultaneously, we have

$$\begin{cases} x = \mathbf{8750} \\ y = 6500 \\ z = 5000 \end{cases}$$

35. An investment of P 4600 is made at an annual simple interest rate of 6.8%. How much additional money must be invested at an annual simple interest rate of 9% so that the total interest earned is 8% of the total investment?

Solution:

Let: x = amount invested at 9%

Income from 6.8% interest = $4500(0.068) = 312.80$

Total income = $312.80 + 0.09x = 0.08(x + 4600)$

$$x = \mathbf{P 5520}$$

36. A and B submitted separate proposals for the construction of a bridge, A offering a lower price for the winning bid. Had A and B reduce their bid prices by 5% and 10% respectively, A would still have won the bid, but the difference in their bids would have been reduced by 3000 pesos. If the sum total of the bids is 90000 pesos, what is the bid of A?

Solution:

Let:

x = bid of A; y = bid of B

Original difference in their bid = $y - x$

Difference when their bid priced reduced = $0.9y - 0.095x$

Since the bid will reduce by 3000, hence:

$$0.9y - 0.095x = y - x - 3000$$

Simplifying,

$$0.05x - 0.1y = -3000 \rightarrow \text{equation 1}$$

And the sum of their bid is 90000

$$x + y = 90000 \rightarrow \text{equation 2}$$

Solving simultaneously,

$$x = 40000; y = 50000$$

Non-Linear Problems

37. Two pipes are used to fill a water storage tank. The first pipe can fill the tank in 4 hours, and the two pipes together can fill the tank in 2 hours less time than the second pipe alone. How long would it take for the second pipe to fill the tank?

Solution:

Time for the first pipe to fill the tank alone = 4 hours

Let: t = time for the second pipe to fill the tank alone

Hence:

$$\text{rate of the first pipe} = \frac{1}{4}$$

$$\text{rate of the second pipe} = \frac{1}{t}$$

Working together, the equation becomes:

$$\left(\frac{1}{4} + \frac{1}{t}\right)(t - 2) = 1$$

$$\mathbf{t = 4 \text{ hours}}$$

38. Scott and Laura have both invested some money. Scott invested P3,000 more than Laura and at a 2% higher interest rate. If Scott received P800 annual interest and Laura received P400, how much did Scott invest?

Solution:

Let:

x = amount Laura invested

$x + 3000$ = amount Scott invested

y = rate of interest for Laura's investment

$y + 0.02$ = rate of interest for Scott's investment

$$xy = 400 \rightarrow \text{income of Laura}$$

$$(x + 3000)(y + 0.02) = 800 \rightarrow \text{amount of Scott}$$

From the first equation:

$$y = \frac{400}{x}$$

$$(x + 3000)\left(\frac{400}{x} + 0.02\right) = 800$$

$$x = 5000 \rightarrow \text{amount Laura invested}$$

$$x + 3000 = \mathbf{P 8000} \rightarrow \text{amount Scott invested}$$

39. One of the important cities of the ancient world was Babylon. Greek historians wrote that the city was square-shaped. Its area numerically exceeded its perimeter by about 124. Find its dimensions in miles.

Solution:

Let: x = side of the city

$$x^2 = 4x + 124$$

$$\mathbf{x = 13.31 \text{ miles}}$$

40. Small fishing boat heads to a point 24 km downriver and then returns. The current moves at the rate of 3 kilometers per hour. If the trip up and back takes 6 hours and the boat keeps a constant speed relative to the water, what is the speed of the boat?

Solution:

Let:

V_B = speed of boat in still water

Distance traveled = 24 km

Total speed downstream = $V_B + 3$

Total speed upstream = $V_B - 3$

Let: t_1 = time of travel upstream

$$(V_B - 3)t_1 = 24$$
$$t_1 = \frac{24}{V_B - 3}$$

Let: t_2 = time of travel downstream

$$(V_B + 3)t_2 = 24$$
$$t_2 = \frac{24}{V_B + 3}$$

The total time of travel is:

$$t_1 + t_2 = 6 \text{ hours}$$

Substituting:

$$\frac{24}{V_B - 3} + \frac{24}{V_B + 3} = 6$$
$$\mathbf{V_B = 9 \text{ kph}}$$

ALGEBRA: POLYNOMIALS, PARTIAL FRACTIONS AND INEQUALITIES

- What will be the result when the polynomial $x^8 + x^7 + 3x^4 - 1$ is divided by $x^4 - 3x^3 + 4x + 1$?
 - $x^4 + 7x^3 + 2x^2 + 11x + 81$; Rem. $14x^3 - 140x^2 - 360x - 83$
 - $x^4 + x^3 + x^2 + 2x + 82$; Rem. $111x^3 - 150x^2 - 30x - 80$
 - $x^4 + 4x^3 + 12x^2 + 32x + 82$; Rem. $194x^3 - 140x^2 - 360x - 83$
 - $x^4 + 5x^3 + 8x^2 + 2x + 2$; Rem. $360x - 83$
- What is the result when the polynomial $2x^4 - 6x^3 + 7x^2 - 5x + 1$ is divided by $x + 2$?
 - $2x^3 - 10x^2 + 27x - 59$; Remainder 119
 - $2x^3 - 2x^2 + 3x + 1$; Remainder 3
 - $2x^3 + 10x^2 - 27 + 59$; Remainder - 119
 - $2x^3 + 2x^2 - 3x - 1$; Remainder - 3
- What is the result when the polynomial $2x^4 - 3x^3 + 8x^2 - 5x + 1$ is divided by $3x + 2$?
 - $\frac{2}{3}x^3 - \frac{13}{9}x^2 - \frac{98}{27}x + \frac{331}{81}$; Remainder $\frac{743}{243}$
 - $\frac{2}{3}x^3 - \frac{13}{9}x^2 + \frac{98}{27}x - \frac{331}{81}$; Remainder $\frac{743}{243}$
 - $\frac{2}{3}x^3 - \frac{13}{9}x^2 + \frac{98}{27}x - \frac{331}{81}$; Remainder $-\frac{743}{243}$
 - $\frac{2}{3}x^3 - \frac{13}{9}x^2 + \frac{98}{27}x - \frac{331}{81}$; Remainder $-\frac{743}{243}$
- What is the remainder when $2x^4 + 9x^3 - 5x^2 - 5x + 7$ is divided by $x + i$?

A. $14 + 14i$	C. $-14 + 4i$
B. $4 + 14i$	D. $14 + 14i$
- Find the cubic Equation whose roots are 0, 1, and 2.

A. $x^3 + 3x^2 + 2x$	C. $x^3 - 3x^2 - 2x$
B. $x^3 - 3x^2 + 2x$	D. $x^3 + 3x^2 - 2x$
- Find the quadric equation with roots $i, -i, 1 + i$, and $1 - i$.

A. $x^4 - x^3 + 2x^2 - 2x + 2$	C. $2x^4 - 2x^3 + 3x^2 - 2x + 2$
B. $x^4 - 2x^3 + 3x^2 - x + 2$	D. $x^4 - 2x^3 + 3x^2 - 2x + 2$
- The length of a FedEx 25-kg box is 7 inches more than its height. The width of the box is 4 inches more than its height. If the volume of the box is 4,420 cubic inches, find the height of the box.

A. 12 inches	C. 15 inches
B. 13 inches	D. 10 inches
- Find the quadratic equation whose sum of the roots is 5 and whose product of the root equal to 6.

A. $x^2 - 5x + 6$	C. $x^2 + 5x + 6$
B. $x^2 - 5x - 6$	D. $x^2 + 5x - 6$
- Let $f(x) = x^5 + ax^4 - 3x^3 + bx - 4$. If $f(x)$ is divided by $x + 2$, the remainder is 10, when divided by $x + 4$ the remainder is -344. What is the value of a ?

A. 5	C. 2
B. -2	D. -5

10. Determine the value of A and B:

$$\frac{9x + 2}{(x + 2)(3x - 2)} = \frac{A}{x + 2} + \frac{B}{3x - 2}$$

A. $A = 2; B = 3$

C. $A = 3; B = 2$

B. $A = -2; B = 3$

D. $A = 3; B = -2$

11. Determine the value of A, B and C:

$$\frac{5x^2 - 25x + 8}{(3x + 2)(x - 3)^2} = \frac{A}{3x + 2} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2}$$

A. $A = 1; B = 2; C = -2$

C. $A = 2; B = 1; C = -2$

B. $A = 2; B = -1; C = 2$

D. $A = 1; B = -2; C = 2$

12. Determine the value of A, B and C:

$$\frac{2x^2 + x + 1}{x^3 + x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

A. $A = 1; B = -1; C = 1$

C. $A = 1; B = 1; C = -1$

B. $A = 1; B = 1; C = 1$

D. $A = 1; B = -1; C = -1$

13. Determine the value of A, B, C, and D:

$$\frac{3x^2 + 5x + 5}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}$$

A. $A = 5; B = 0; C = 2; D = 3$

C. $A = 0; B = 3; C = 5; D = 2$

B. $A = 3; B = 2; C = 3; D = 0$

D. $A = 2; B = 5; C = 0; D = 5$

14. Determine the value of A, B, C, D and E:

$$\frac{4x^4 - 7x^3 + 5x^2 - x + 1}{(2x - 1)(x^2 - x + 1)^2} = \frac{A}{2x - 1} + \frac{Bx + C}{x^2 - x + 1} + \frac{Dx + E}{(x^2 - x + 1)^2}$$

A. $A = 2; B = 0; C = 1; D = -2; E = 1$

B. $A = 2; B = 2; C = 0; D = -2; E = 1$

C. $A = 2; B = -1; C = 0; D = -2; E = 1$

D. $A = 2; B = 1; C = 0; D = -2; E = 1$

15. Solve for x:

$$5(x - 4) > 25$$

A. $x < 9$

C. $x > 9$

B. $x > 1$

D. $x < 1$

16. Solve for x:

$$-4(x + 3) \geq 16$$

A. $x \leq -7$

C. $x \geq -7$

B. $x \leq 7$

D. $x \geq 7$

17. Solve for x:

$$4 < 2x - 8 \leq 10$$

A. $6 < x \leq 9$

C. $6 \leq x < 9$

B. $-9 \leq x < -6$

D. $-9 < x \leq 6$

18. Solve for x:

$$2 + x < 3x - 2 < 5x + 2$$

A. $x > 2$

C. $x > -2$

B. $x < 2$

D. $x < -2$

19. Solve for x:

$$x^2 - 13x + 12 \leq 0$$

A. $1 \geq x \geq 12$

C. $-1 \leq x \leq 12$

B. $-12 \leq x \leq -1$

D. $1 \leq x \leq 12$

20. Solve for x:

$$\frac{x+3}{x-2} < 0$$

A. $-2 < x < 3$

C. $-3 < x \leq 2$

B. $-3 < x < 2$

D. $2 < x < 3$

21. Solve for x:

$$\frac{x^2 + 10x + 25}{x^2 - x - 12} \leq 0$$

A. $-3 < x < 4$

C. $-4 < x < -3$

B. $x = -5, -3 < x < 4$

D. $x = -5, -4 < x < 3$

22. Solve for x:

$$|x - 2| < 7$$

A. $-5 < x < 7$

C. $-5 < x < 9$

B. $-7 < x < 9$

D. $-7 < x < 7$

23. Solve for x:

$$\left| \frac{2x+3}{2} \right| + 7 \geq 12$$

A. $x \geq -\frac{13}{2} \cup x \geq \frac{7}{2}$

C. $x \leq \frac{13}{2} \cup x \geq -\frac{7}{2}$

B. $x \leq \frac{13}{2} \cup x \geq \frac{7}{2}$

D. $x \leq -\frac{13}{2} \cup x \geq \frac{7}{2}$

24. Solve for x:

$$0 < |x - 5| \leq 3$$

A. $[2,5) \cup (5,8]$

C. $[-2,5) \cup (5,8]$

B. $[2,5] \cup [5,8]$

D. $[-2,5] \cup [5,8]$

25. Solve for x:

$$|x + 2| > |x + 1|$$

A. $x > \frac{3}{2}$

C. $x < \frac{3}{2}$

B. $x > -\frac{3}{2}$

D. $x < -\frac{3}{2}$

ALGEBRA: POLYNOMIALS, PARTIAL FRACTIONS AND INEQUALITIES

SOLUTIONS

1. What will be the result when the polynomial $x^8 + x^7 + 3x^4 - 1$ is divided by $x^4 - 3x^3 + 4x + 1$?

First Solution:

Conventional:

		x^4	$+4x^3$	$+12x^2$	$+32x$	$+82$
$x^4 - 3x^3 + 4x + 1$	$x^8 + x^7$	$+3x^4$				-1
	$x^8 - 3x^7$	$+4x^5$	$+x^4$			
	$4x^7$	$-4x^5$	$+2x^4$			-1
	$4x^7 - 12x^6$	$+16x^4$	$+4x^3$			
	$12x^6$	$-4x^5$	$-14x^4$	$-4x^3$		-1
	$12x^6 - 36x^5$	$+48x^3$	$+12x^2$			
	$32x^5$	$-14x^4$	$-52x^3$	$-12x^2$		-1
	$32x^5 - 96x^4$	$+128x^2$	$+32x$			
	$82x^4$	$-52x^3$	$-140x^2$	$-32x$		-1
	$82x^4 - 246x^3$	$+328x$	$+82$			
	$194x^3$	$-140x^2$	$-360x$	-83		

Hence, the answer is:

$$x^4 + 4x^3 + 12x^2 + 32x + 82$$

And the remainder is:

$$194x^3 - 140x^2 - 360x - 83$$

Second Solution:

Using the form:

$$\frac{p(x)}{g(x)} = Q(x) + \frac{r(x)}{g(x)}$$

Where:

$p(x)$ = dividend

$g(x)$ = divisor

$Q(x)$ = quotient

$r(x)$ = remainder

$$p(x) = Q(x)g(x) + r(x)$$

We can use the choices to check which among them satisfies equality for a certain value:

$$p(x) = x^8 + x^7 + 3x^4 - 1$$

$$g(x) = x^4 - 3x^3 + 4x + 1$$

For choice a:

$$Q(x) = x^4 + 7x^3 + 2x^2 + 11x + 81$$

$$r(x) = 194x^3 - 140x^2 - 360x - 83$$

Try a test value, say $x = 3$:

Go to MODE 1 and type:

$$x^8 + x^7 + 3x^4 - 1$$

And then CALC $x = 3$ then press =

Then the answer will be 8990.

Hence,

$$p(x) = 8990$$

For $g(x)$, go to MODE 1 and type:

$$x^4 - 3x^3 + 4x + 1$$

Then press CALC $x = 3$:

Then the answer will be 13.

Hence,

$$g(x) = 13$$

For $Q(x)$, go to MODE 1 and type:

$$x^4 + 7x^3 + 2x^2 + 11x + 81$$

Again, press CALC $x = 3$:

The answer will be 402.

Hence,

$$Q(x) = 402$$

For $r(x)$, go to MODE 1 and type:

$$194x^3 - 140x^2 - 360x - 83$$

Again, press CALC $x = 3$:

The answer will be -2045.

Hence,

$$r(x) = -2045$$

And then check:

$$p(x) = Q(x)g(x) + r(x)$$

$$8990 = (402)(13) + (-2045)$$

$$8990 \neq 3181$$

Therefore, a is not the answer.

For convenience, adopt $x = 3$ for all trial value so that $p(x) = 8990$ and $g(x) = 13$ are the same for all choices.

For choice b:

$$p(x) = 8990, g(x) = 13$$

$$Q(x) = x^4 + x^3 + x^2 + 2x + 82 \text{ at } x = 3$$

$$Q(x) = 205$$

$$r(x) = 111x^3 - 150x^2 - 30x - 80 \text{ at } x = 3$$

$$r(x) = 1477$$

$$p(x) = Q(x)g(x) + r(x)$$

$$8990 = (205)(13) + 1477$$

$$8990 \neq 4142$$

Therefore, b is not the answer.

For choice c:

$$p(x) = 8990, g(x) = 13$$

$$Q(x) = x^4 + 4x^3 + 12x^2 + 32x + 82 \text{ at } x = 3$$

$$Q(x) = 475$$

$$r(x) = 194x^3 - 140x^2 - 360x - 83 \text{ at } x = 3$$

$$r(x) = 2815$$

$$p(x) = Q(x)g(x) + r(x)$$

$$8990 = (475)(13) + 2815$$

$$8990 = 8990$$

Therefore, c is true.

For choice d:

$$\begin{aligned}
 p(x) &= 8990, g(x) = 13 \\
 Q(x) &= x^4 + 5x^3 + 8x^2 + 2x + 2 \text{ at } x = 3 \\
 Q(x) &= 296 \\
 r(x) &= 360x - 83 \text{ at } x = 3 \\
 r(x) &= 997 \\
 p(x) &= Q(x)g(x) + r(x) \\
 8990 &= (296)(13) + 997 \\
 8990 &\neq 4845
 \end{aligned}$$

Therefore, d is not the answer.

Since c is the only choice that satisfies the equality and the remaining 3 choices are false then our answer is letter c.

Note: In doing this technique be sure that only one choice satisfies the equality, if more than one choice satisfies, try to use another trial value.

2. What is the result when the polynomial $2x^4 - 6x^3 + 7x^2 - 5x + 1$ is divided by $x + 2$?

Solution:

Use synthetic division:

$$\begin{array}{r|rrrrr}
 -2 & 2 & -6 & 7 & -5 & 1 \\
 & & -4 & 20 & -54 & 118 \\
 \hline
 & 2 & -10 & 27 & -59 & 119
 \end{array}$$

The answer is:

$$\frac{2x^4 - 6x^3 + 7x^2 - 5x + 1}{x + 2} = 2x^3 + 10x^2 + 27x - 59 + \frac{119}{x + 2}$$

3. What is the result when the polynomial $2x^4 - 3x^3 + 8x^2 - 5x + 1$ is divided by $3x + 2$?

Solution:

Before performing synthetic division, divide both numerator and denominator by 3:

$$\frac{(2x^4 - 3x^3 + 8x^2 - 5x + 1) \left(\frac{1}{3}\right)}{(3x + 2) \left(\frac{1}{3}\right)} = \frac{\frac{2}{3}x^4 - x^3 + \frac{8}{3}x^2 - \frac{5}{3}x + \frac{1}{3}}{x + \frac{2}{3}}$$

By synthetic division:

$$\begin{array}{r|rrrrr}
 -\frac{2}{3} & \frac{2}{3} & -1 & \frac{8}{3} & -\frac{5}{3} & \frac{1}{3} \\
 & & -\frac{4}{9} & \frac{26}{27} & -\frac{196}{81} & \frac{662}{243} \\
 \hline
 & \frac{2}{3} & -\frac{13}{9} & \frac{98}{27} & -\frac{331}{81} & \frac{743}{243}
 \end{array}$$

Therefore, the answer is:

$$\frac{\frac{2}{3}x^4 - x^3 + \frac{8}{3}x^2 - \frac{5}{3}x + \frac{1}{3}}{x + \frac{2}{3}} = \frac{2}{3}x^3 - \frac{13}{9}x^2 + \frac{98}{27}x - \frac{331}{81} + \frac{743}{243} \frac{1}{x + \frac{2}{3}}$$

4. What is the remainder when $2x^4 + 9x^3 - 5x^2 - 5x + 7$ is divided by $x + i$?

Solution:

Using Remainder Theorem:

Go to MODE 2 (Complex Mode) to evaluate:

$$2x^4 + 9x^3 - 5x^2 - 5x + 7 \text{ when } x = -i$$

Because the calculator cannot evaluate x^4 , the equation must be typed this way:

$$2x^3x + 9x^3 - 5x^2 - 5x + 7$$

Hence,

$$2x^3x + 9x^3 - 5x^2 - 5x + 7 \text{ at } x = -i$$

The answer is **14 + 14i**.

5. Find the cubic Equation whose roots are 0, 1, and 2.

Solution:

Using factor theorem:

$$x(x - 1)(x - 2) = 0$$

$$(x^2 - x)(x - 2) = 0$$

$$x^3 - 2x^2 - x^2 + 2x = 0$$

$$\mathbf{x^3 - 3x^2 + 2x = 0}$$

6. Find the quadric equation with roots $i, -i, 1 + i$, and $1 - i$.

First Solution:

Using factor theorem:

$$(x - i)(x + i)(x - (1 + i))(x - (1 - i)) = 0$$

$$(x^2 - i^2)(x^2 - (1 - i)x - (1 + i)x + (1 + i)(1 - i)) = 0$$

$$(x^2 + 1)(x^2 - x + ix - x - ix + 1 - i + i - i^2) = 0$$

$$(x^2 + 1)(x^2 - 2x + 2) = 0$$

$$x^4 - 2x^3 + 2x^2 + x^2 - 2x + 2 = 0$$

$$\mathbf{x^4 - 2x^3 + 3x^2 - 2x + 2 = 0}$$

Second Solution:

Try the choices using remainder theorem to check if the given roots are really roots of an equation.

Using MODE 2, type and press CALC for the values of x :

Choice a:

$$x^3x - x^3 + 2x^2 - 2x + 2 \text{ when } x = i$$

The value is $1 - i$ (remainder not zero)

Therefore, a is not an answer.

Choice b:

$$2x^3x - 2x^3 + 3x^2 - 2x + 2 \text{ when } x = i$$

The value is 1 (remainder not zero)

Therefore, b is not an answer.

Choice c:

$$x^3x - 2x^3 + 3x^2 - x + 2 \text{ when } x = i$$

The value is i (remainder not zero)

Therefore, c is not an answer.

Choice d:

$$x^3x - 2x^3 + 3x^2 - x + 2 \text{ when } x = i, \text{ value is } 0 \rightarrow \text{root}$$

$$x^3x - 2x^3 + 3x^2 - x + 2 \text{ when } x = -i, \text{ value is } 0 \rightarrow \text{root}$$

$$x^3x - 2x^3 + 3x^2 - x + 2 \text{ when } x = 1 + i, \text{ the value is } 0 \rightarrow \text{root}$$

$$x^3x - 2x^3 + 3x^2 - x + 2 \text{ when } x = 1 - i, \text{ the value is } 0 \rightarrow \text{root}$$

Since all given roots were roots of equation in choice d, therefore the answer is **d**.

7. The length of a FedEx 25-kg box is 7 inches more than its height. The width of the box is 4 inches more than its height. If the volume of the box is 4,420 cubic inches, find the height of the box.

Solution:

Let:

x = height of the box

Length = $x + 7$

Width = $x + 4$

$$V = lwh$$

$$440 = x(x + 7)(x + 4)$$

Solving using Shift-Solve: $x = \mathbf{13 \text{ inches}}$.

8. Find the quadratic equation whose sum of the roots is 5 and whose product of the root equal to 6.

Solution:

$$x^2 - (r_1 + r_2)x + r_1r_2 = 0$$

$$\mathbf{x^2 - 5x + 6 = 0}$$

9. Let $f(x) = x^5 + ax^4 - 3x^3 + bx - 4$. If $f(x)$ is divided by $x + 2$, the remainder is 10, when divided by $x + 4$ the remainder is -344. What is the value of a ?

Solution:

By remainder theorem:

$$f(-2) = 10$$

$$(-2)^5 + a(-2)^4 - 3(-2)^3 + b(-2) - 4 = 10$$

$$16a - 2b - 12 = 10$$

$$16a - 2b = 22 \rightarrow \text{equation 1}$$

$$f(-4) = -344$$

$$(-4)^5 + a(-4)^4 - 3(-4)^3 + b(-4) - 4 = -344$$

$$256a - 4b - 836 = -344$$

$$256a - 4b = 492 \rightarrow \text{equation 2}$$

Solving simultaneously, go to MODE 5-1:

$$\mathbf{a = 2}$$

$$\mathbf{b = 5}$$

10. Determine the value of A and B:

$$\frac{9x + 2}{(x + 2)(3x - 2)} = \frac{A}{x + 2} + \frac{B}{3x - 2}$$

Solution:

To solve for A, multiply both sides of the equation by $(x + 2)$:

$$\frac{9x + 2}{3x - 2} = A + B \times \left(\frac{x + 2}{3x - 2} \right)$$

Substitute $x = -2$ to eliminate B:

$$\frac{9(-2) + 2}{3(-2) - 2} = A$$

$$\mathbf{A = 2}$$

To solve for B, multiply both sides of the equation by $(3x - 2)$:

$$\frac{9x + 2}{x + 2} = A \times \left(\frac{3x - 2}{x + 2} \right) + B$$

Substitute $x = 2/3$ to eliminate A:

$$\frac{9\left(\frac{2}{3}\right) + 2}{\frac{2}{3} + 2} = B$$

$$\mathbf{B = 3}$$

11. Determine the value of A, B and C:

$$\frac{5x^2 - 25x + 8}{(3x + 2)(x - 3)^2} = \frac{A}{3x + 2} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2}$$

Solution:

To solve for A, multiply both sides of the equation by $(3x + 2)$:

$$\frac{5x^2 - 25x + 8}{(x - 3)^2} = A + \frac{B(3x + 2)}{x - 3} + \frac{C(3x + 2)}{(x - 3)^2}$$

Substitute $x = -2/3$ to eliminate B and C:

$$\frac{5\left(-\frac{2}{3}\right)^2 - 25\left(-\frac{2}{3}\right) + 8}{\left(-\frac{2}{3} - 3\right)^2} = A$$

$$\mathbf{A = 2}$$

To solve for C, multiply both sides of the equation by $(x - 3)^2$:

$$\frac{5x^2 - 25x + 8}{(3x + 2)} = \frac{A(x - 3)^2}{3x + 2} + B(x - 3) + C$$

Substitute $x = 3$ to eliminate A and B:

$$\frac{5(3)^2 - 25(3) + 8}{(3(3) + 2)} = C$$

$$\mathbf{C = -2}$$

To solve for B, let $x = 0$: (you can substitute any value except $-2/3$ and 3)

$$\frac{5(0)^2 - 25(0) + 8}{(3(0) + 2)(0 - 3)^2} = \frac{2}{3(0) + 2} + \frac{B}{0 - 3} - \frac{2}{(0 - 3)^2}$$

$$\frac{4}{9} = 1 - \frac{B}{3} - \frac{2}{9}$$

$$\mathbf{B = 1}$$

12. Determine the value of A, B and C:

$$\frac{2x^2 + x + 1}{x^3 + x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

Solution:

To solve for A, multiply both sides of the equation by x:

$$\frac{2x^2 + x + 1}{x^2 + 1} = A + \frac{(Bx + C)x}{x^2 + 1}$$

Substitute $x = 0$ to eliminate B and C:

$$\frac{2(0)^2 + 0 + 1}{0^2 + 1} = A$$

$$\mathbf{A = 1}$$

To solve for B and C, multiply both sides of the equation by $(x^2 + 1)$:

$$\frac{2x^2 + x + 1}{x} = \frac{A(x^2 + 1)}{x} + Bx + C$$

Substitute $x = i$ to eliminate A:

$$\frac{2i^2 + i + 1}{i} = Bi + C$$

$$1 + i = Bi + C$$

Equate real to real, and imaginary to imaginary:

$$Bi = i$$

$$\mathbf{B = 1}$$

$$\mathbf{C = 1}$$

13. Determine the value of A, B, C, and D:

$$\frac{3x^2 + 5x + 5}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}$$

To solve for C and D, multiply both sides of the equation by $(x^2 + 1)^2$:

$$3x^2 + 5x + 5 = (Ax + B)(x^2 + 1) + Cx + D$$

Substitute $x = i$ to eliminate A and B:

$$3i^2 + 5i + 5 = Ci + D$$

$$2 + 5i = Ci + D$$

Equate real to real, imaginary to imaginary:

$$Ci = 5i$$

$$\mathbf{C = 5}$$

$$\mathbf{D = 2}$$

To solve for B, let $x = 0$ to eliminate A:

$$\frac{3(0)^2 + 5(0) + 5}{(0^2 + 1)^2} = \frac{A(0) + B}{0^2 + 1} + \frac{5(0) + 2}{(0^2 + 1)^2}$$

$$5 = B + 2$$

$$\mathbf{B = 3}$$

To solve for A, let $x = 1$:

$$\frac{3(1)^2 + 5(1) + 5}{(1^2 + 1)^2} = \frac{A(1) + 3}{1^2 + 1} + \frac{5(1) + 2}{(1^2 + 1)^2}$$

$$\frac{13}{4} = \frac{A + 3}{2} + \frac{7}{4}$$

$$\mathbf{A = 0}$$

14. Determine the value of A, B, C, D and E:

$$\frac{4x^4 - 7x^3 + 5x^2 - x + 1}{(2x - 1)(x^2 - x + 1)^2} = \frac{A}{2x - 1} + \frac{Bx + C}{x^2 - x + 1} + \frac{Dx + E}{(x^2 - x + 1)^2}$$

Solution:

To solve for A, multiply both sides of the equation by $(2x - 1)$:

$$\frac{4x^4 - 7x^3 + 5x^2 - x + 1}{(x^2 - x + 1)^2} = A + \frac{(Bx + C)(2x - 1)}{x^2 - x + 1} + \frac{(Dx + E)(2x - 1)}{(x^2 - x + 1)^2}$$

Substitute $x = 1/2$ to eliminate B, C, D, and E:

$$\frac{4\left(\frac{1}{2}\right)^4 - 7\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^2 - \frac{1}{2} + 1}{\left(\left(\frac{1}{2}\right)^2 - \frac{1}{2} + 1\right)^2} = A$$

$$\mathbf{A = 2}$$

To solve for D and E, multiply both sides of the equation by $(x^2 - x + 1)^2$:

$$\frac{4x^4 - 7x^3 + 5x^2 - x + 1}{(2x - 1)} = \frac{A(x^2 - x + 1)^2}{2x - 1} + (Bx + C)(x^2 - x + 1) + Dx + E$$

Using MODE 5-3, solve for the roots of $x^2 - x + 1$:

$$x = \frac{1}{2} + \frac{\sqrt{3}}{2}i; \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Substitute $x = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ to eliminate A, B, and C: (Note: x^4 should write as either of x^3x or x^2x^2 and use MODE 2)

$$\frac{4x^4 - 7x^3 + 5x^2 - x + 1}{(2x - 1)} = \frac{A(x^2 - x + 1)^2}{2x - 1} + (Bx + C)(x^2 - x + 1) + Dx + E$$

$$0 - \sqrt{3}i = D\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + E$$

Equate real to real, and imaginary to imaginary:

$$0 = \frac{D}{2} + E$$

$$-\sqrt{3}i = \frac{D\sqrt{3}}{2}i$$

$$\mathbf{D = -2}$$

$$\mathbf{E = 1}$$

To solve for C, let $x = 0$:

$$\frac{4(0)^4 - 7(0)^3 + 5(0)^2 - 0 + 1}{(2(0) - 1)(0^2 - 0 + 1)^2} = \frac{2}{2(0) - 1} + \frac{B(0) + C}{0^2 - 0 + 1} + \frac{-2(0) + 1}{(0^2 - 0 + 1)^2}$$

$$-1 = -2 + C + 1$$

$$\mathbf{C = 0}$$

To solve for B, let $x = 1$:

$$\frac{4(1)^4 - 7(1)^3 + 5(1)^2 - 1 + 1}{(2(1) - 1)(1^2 - 1 + 1)^2} = \frac{2}{2(1) - 1} + \frac{B(1) + 0}{1^2 - 1 + 1} + \frac{-2(1) + 1}{(1^2 - 1 + 1)^2}$$

$$2 = 2 + B - 1$$

$$\mathbf{B = 1}$$

15. Solve for x:

$$5(x - 4) > 25$$

Solution:

$$\begin{aligned} 5(x - 4) &> 25 \\ (x - 4) &> 5 \\ \mathbf{x} &> \mathbf{9} \end{aligned}$$

16. Solve for x:

$$-4(x + 3) \geq 16$$

Solution:

$$\begin{aligned} -4(x + 3) &\geq 16 \\ (x + 3) &\leq -4 \\ \mathbf{x} &\leq \mathbf{-7} \end{aligned}$$

17. Solve for x:

$$4 < 2x - 8 \leq 10$$

Solution:

$$\left(\begin{array}{l|l} 4 < 2x - 8 & 2x - 8 \leq 10 \\ 12 < 2x & 2x \leq 18 \\ 6 < x & x \leq 9 \\ x > 6 & \end{array} \right)$$

Getting the intersection: $\mathbf{6 < x \leq 9}$

18. Solve for x:

$$2 + x < 3x - 2 < 5x + 2$$

Solution:

$$\begin{aligned} 2 + x < 3x - 2 < 5x + 2 \\ \left(\begin{array}{l|l} 2 + x < 3x - 2 & 3x - 2 < 5x + 2 \\ 4 < 2x & -4 < 2x \\ 2 < x & -2 < x \end{array} \right) \\ 2 < x \text{ and } -2 < x \end{aligned}$$

Therefore, $2 < x \cap -2 < x \rightarrow \mathbf{x > 2}$.

19. Solve for x:

$$x^2 - 13x + 12 \leq 0$$

Solution:

For inequalities of polynomial equation, rewrite the inequalities, such that the right side is zero.

$$x^2 - 13x + 12 \leq 0$$

For polynomials, get the zero of polynomial

$$x^2 - 13x + 12 = 0$$

$$(x - 1)(x - 12) = 0$$

Thus, the roots are $x = 1$ and $x = 12$.

Interval	Test Value	Result	Remarks
$(-\infty, 1]$	Say $x = 0$	$12 > 0$	All numbers in this interval is not a solution
$[1, 12]$	Say $x = 3$	$-18 \leq 0$	All number in this interval is a solution
$[12, \infty)$	Say $x = 13$	$12 \geq 0$	All number in this interval is not a solution

Therefore,

$$1 \leq x \leq 12$$

20. Solve for x:

$$\frac{x + 3}{x - 2} < 0$$

Solution:

For rational inequality, rewrite the equation such that the right side is zero.

$$\frac{x + 3}{x - 2} < 0$$

For rational inequality, get the zero and the asymptote.

Thus, getting the zero:

$$\begin{aligned} \frac{x + 3}{x - 2} &= 0 \\ x - 2 &= -3 \end{aligned}$$

And getting the asymptote;

$$\begin{aligned} x - 2 &= 0 \\ x &= 2 \end{aligned}$$

Interval	Test Value	Result	Remarks
$(-\infty, -3)$	Say $x = -5$	$\frac{2}{7} > 0$	All numbers in this interval is not a solution
$(-3, 2)$	Say $x = 0$	$-\frac{3}{2} < 0$	All number in this interval is a solution
$[2, \infty)$	Say $x = 3$	$6 > 0$	All number in this interval is not a solution

Therefore,

$$-3 < x < 2$$

21. Solve for x:

$$\frac{x^2 + 10x + 25}{x^2 - x - 12} \leq 0$$

Solution:

$$\frac{x^2 + 10x + 25}{x^2 - x - 12} \leq 0$$

Solving for the zeroes,

$$\begin{aligned} \frac{x^2 + 10x + 25}{x^2 - x - 12} &= 0 \\ x^2 + 10x + 25 &= 0 \\ x &= -5 \end{aligned}$$

Solving for asymptotes,

$$\begin{aligned} x^2 - x - 12 &= 0 \\ x &= -3, x = 4 \end{aligned}$$

Interval	Test Value	Result	Remarks
$(-\infty, -5]$	Say $x = -6$	$\frac{1}{30} > 0$	All numbers in this interval is not a solution
$(-5, -3)$	Say $x = -4$	$\frac{1}{8} > 0$	All number in this interval is not a solution
$(-3, 4)$	Say $x = 0$	$-\frac{25}{12} \leq 0$	All number in this interval is a solution
$(4, \infty)$	Say $x = 5$	$\frac{25}{2} > 0$	All number in this interval is not a solution

Therefore,

$$x = -5, -3 < x < 4$$

22.Solve for x:

$$|x - 2| < 7$$

Solution:

$$|x - 2| < 7$$

It is equivalent to:

$$-7 < x - 2 < 7$$

$$-5 < x < 9$$

23.Solve for x:

$$\left| \frac{2x + 3}{2} \right| + 7 \geq 12$$

Solution:

$$\left| \frac{2x + 3}{2} \right| + 7 \geq 12$$

$$\left| \frac{2x + 3}{2} \right| \geq 5$$

$$\left\{ \begin{array}{l} \frac{2x + 3}{2} \leq -5 \\ 2x + 3 \leq -10 \\ 2x \leq -13 \\ x \leq -\frac{13}{2} \end{array} \right\} \left\{ \begin{array}{l} \frac{2x + 3}{2} \geq 5 \\ 2x + 3 \geq -10 \\ 2x \geq 7 \\ x \geq \frac{7}{2} \end{array} \right\}$$

Therefore,

$$x \leq -\frac{13}{2} \cup x \geq \frac{7}{2}$$

24.Solve for x:

$$0 < |x - 5| \leq 3$$

Solution:

The inequality $0 < |x - 5| \leq 3$ consists of two consecutive inequalities that can be solved separately. The solution will be the intersection of the inequalities.

$$0 < |x - 5| \text{ and } |x - 5| \leq 3$$

The inequality $0 < |x - 5|$ is true for all numbers except 5.

The inequality $|x - 5| \leq 3$ is equivalent to the inequality,

$$-3 \leq x - 5 \leq 3$$

$$2 \leq x \leq 8$$

The solution set is the intersection of these two solutions, which is the interval $[2, 8]$ except 5. This is the union of the two intervals **$[2, 5)$ and $(5, 8]$** .

25.Solve for x:

$$|x + 2| > |x + 1|$$

Solution:

$$|x + 2| > |x + 1|$$

Squaring both sides to eliminate the absolute value:

$$x^2 + 4x + 4 > x^2 + 2x + 1$$

$$2x > -3$$

$$x > -\frac{3}{2}$$

ALGEBRA: BINOMIAL EXPANSION

1. Find the 5th term in the expansion of $(2x - 3y^2)^8$.

A. $72576x^4y^8$	C. $90720x^4y^8$
B. $-90720x^4y^8$	D. $-72576x^4y^8$

2. Find the sum of the coefficients in the expansion of $(3x + y - 2z)^5$.

A. 32	C. -32
B. -7776	D. 7776

3. Find the sum of the coefficients in the expansion of $(3x^2 + y - 2)^4$.

A. 16	C. 32
B. 0	D. -16

4. Find the term involving x^9y^6 in the expansion of $(x^3 + 2y^2)^6$.

A. $160y^6$	C. $-160y^6$
B. $160x^9y^6$	D. $-160x^9y^6$

5. Find the term involving y^2z in the expansion of $(3x + y^2 - z)^3$.

A. $18x^2y^2z$	C. $-54x^2y^2z$
B. $-18xy^2z$	D. $-54xy^2z$

6. Find the constant term in the expression $\left(x - \frac{2}{x}\right)^4$.

A. 16	C. -24
B. -16	D. 24

7. The second term in the expansion of $(4x - y)^n$ is Cx^4y . Find C.

A. 120	C. -560
B. 2560	D. -1280

8. In the expansion of $(Ax + By)^5$, find the value(s) of A if the sum of all numerical coefficients is 32 and the product of A and B is -24.

A. -4 and 6	C. 4
B. 6 and 4	D. -8

9. Find the middle term in the expansion of $(x^6 - y^2)^{30}$.

A. $155117520x^{90}y^{30}$	C. $-155117520x^{90}y^{30}$
B. $-145422675x^{90}y^{30}$	D. $145422675x^{90}y^{30}$

10. In the expansion of $(Ax + By)^4$, find the value(s) of A if the sum of all numerical coefficients is 625 and the coefficient of the middle term is 216.

A. $\pm 1, 2, 3, \mp 6$	C. $1, \pm 2, \pm 3, 6$
B. $\pm 1, \pm 2, \pm 3, \pm 6$	D. $1, 2, 3, 6$

ALGEBRA: BINOMIAL EXPANSION SOLUTIONS

1. Find the 5th term in the expansion of $(2x - 3y^2)^8$.

Solution:

Using the formula:

$$\text{rth term} = \binom{n}{r-1} (a)^{n-r+1} (b)^{r+1}$$

$$n = 8$$

$$a = 2x$$

$$b = -3y^2$$

$$r = 5$$

We have,

$$\text{5th term} = \binom{8}{5-1} (2x)^{8-5+1} (-3y^2)^{5-1}$$

$$\text{5th term} = \binom{8}{4} (2x)^4 (-3y^2)^4$$

$$\text{5th term} = 70(16x^4)(81y^8)$$

$$\text{5th term} = \mathbf{90720x^4y^8}$$

2. Find the sum of the coefficients in the expansion of $(3x + y - 2z)^5$.

Solution:

By substituting 1 for all variables:

$$(3x + y - 2z)^5 = [(3)(1) + 1 - 2(z)]^5 = \mathbf{32}$$

3. Find the sum of the coefficients in the expansion of $(3x^2 + y - 2)^4$.

Solution:

By substituting 1 for all variables:

Note: The last term in the expansion of this term is a constant term; therefore, you will subtract it to the value computed. Thus,

$$(2x^2 + y - 2)^4 = [(3)(1) + 1 - 2]^4 - (-2)^4 = \mathbf{0}$$

4. Find the term involving x^9 in the expansion of $(x^3 + 2y^2)^6$.

Solution:

Using the formula:

$$\text{rth term} = \binom{n}{r-1} (a)^{n-r+1} (b)^{r+1}$$

$$n = 6$$

$$a = x^3$$

$$b = 2y^2$$

Equating the exponents of the variables:

$$x^9 = (x^3)^{6-r+1} (2y^2)^{r-1}$$

$$x^9 = (x^3)^{7-r} (2y^2)^{r-1}$$

$$9 = 3(7 - r)$$

Solving for r: $r = 4$

Therefore,

$$\text{4th term} = \binom{6}{4-1} (x^3)^{6-4+1} (2y^2)^{4-1}$$

$$\text{4th term} = \binom{6}{3} (x^9) (2y^2)^3$$

$$\text{4th term} = 20(x^9)(8y^6)$$

$$\text{4th term} = \mathbf{160x^9y^6}$$

5. Find the term involving y^2z in the expansion of $(3x + y^2 - z)^3$.

Solution:

Using the formula:

$$\text{rth term} = \binom{n}{r-1} (a)^{n-r+1} (b)^{r+1}$$

$$n = 3$$

$$a = 3x + y^2$$

$$b = -z$$

Equating the exponents of the variables:

$$y^2z = \binom{3}{r-1} (3x + y^2)^{3-r+1} (-z)^{r-1}$$

Equating exponents of z:

$$1 = r - 1$$

$$r = 2$$

Therefore,

$$\binom{3}{2-1} (3x + y^2)^{3-2+1} (-z)^{2-1}$$

$$\binom{3}{1} (3x + y^2)^2 (-z)^1$$

For $(3x + y^2)^2$:

$$y^2 = \binom{2}{r-1} (3x)^{2-r+1} (y^2)^{r-1}$$

Equating exponents of y:

$$2 = 2(r - 1)$$

$$r = 2$$

Therefore,

$$\binom{2}{2-1} (3x)^{2-2+1} (y^2)^{2-1} = \binom{2}{1} (3x)^1 (y^2)^1$$

$$\binom{2}{2-1} (3x)^{2-2+1} (y^2)^{2-1} = 6xy^2$$

Finally,

$$\binom{3}{2-1} (6xy^2) (-z)^1 = \binom{3}{1} (6xy^2) (-z)$$

$$\binom{3}{2-1} (6xy^2) (-z)^1 = -18xy^2z$$

6. Find the constant term in the expression $\left(x - \frac{2}{x}\right)^4$.

Solution:

$$\text{rth term} = \binom{n}{r-1} (a)^{n-r+1} (b)^{r+1}$$

Equating the exponents of the variables:

$$x^0 = (x)^{4-r+1} (x^{-1})^{r-1}$$

$$x^0 = x^{5-r} \cdot x^{-r+1}$$

$$0 = (5 - r) + (-r + 1)$$

$$0 = 6 - 2r$$

$$r = 3$$

Therefore,

$$\text{3rd term} = \binom{4}{3-1} (x)^{4-3+1} (-2x^{-1})^{3-1} = \binom{4}{2} (x)^2 \left(-\frac{2}{x}\right)^2 = 24$$

7. The second term in the expansion of $(4x - y)^n$ is Cx^4y . Find C.

Solution:

$$\text{rth term} = \binom{n}{r-1} (a)^{n-r+1} (b)^{r+1}$$

$$r = 2$$

$$a = 4x$$

$$b = -y$$

$$\text{2nd term} = \binom{n}{2-1} (4x)^{n-2+1} (-y)^{2+1} = \binom{n}{1} (4x)^{n-1} (-y)^3$$

Equating the exponents of x:

$$x^4 = x^{n-1}$$

$$4 = n - 1$$

$$n = 5$$

Therefore,

$$\binom{5}{2-1} (4x)^{5-2+1} (-y)^{2-1} = \binom{5}{1} (4x)^4 (-y)^1 = -1280x^4y$$

$$\therefore C = -1280$$

8. In the expansion of $(Ax + By)^5$, find the value(s) of A if the sum of all numerical coefficients is 32 and the product of A and B is -24.

Solution:

By substituting 1 to all variables:

$$(Ax + By)^5 = [(A)(1) + (B)(1)]^5 = 32$$

$$(A + B)^5 = 32$$

$$A + B = 2 \rightarrow \text{equation 1}$$

$$A \cdot B = -24$$

$$B = -\frac{24}{A} \rightarrow \text{equation 2}$$

Substitute equation 2 to equation 1:

$$A + \left(-\frac{24}{A}\right) = 2$$

$$A^2 - 24 = 2A$$

$$A^2 - 2A - 24 = 0$$

$$(A - 6)(A + 4) = 0$$

$$\left(\begin{array}{l|l} A - 6 = 0 & A + 4 = 0 \\ A = 6 & A = -4 \\ B = -4 & B = 6 \end{array} \right)$$

Therefore,

$$A = -4 \text{ and } A = 6$$

9. Find the middle term in the expansion of $(x^6 - y^2)^{30}$.

Solution:

Since $n = 30$, therefore there are 31 terms, and the middle term is the 16th term.

$$\text{rth term} = \binom{n}{r-1} (a)^{n-r+1} (b)^{r+1}$$

$$n = 30; r = 16$$

$$a = x^6$$

$$b = -y^2$$

We have,

$$16\text{th term} = \binom{30}{16-1} (x^6)^{30-16+1} (-y^2)^{16-1} = \binom{30}{15} (x^6)^{15} (-y^2)^{15}$$

$$16\text{th term} = \binom{30}{15} (x^{90}) (-y^{30}) = -155117520x^{90}y^{30}$$

10. In the expansion of $(Ax + By)^4$, find the value(s) of A if the sum of all numerical coefficients is 625 and the coefficient of the middle term is 216.

Solution:

By substituting 1 to all variables:

$$(Ax + By)^4 = [(A)(1) + (B)(1)]^4 = 625$$

$$(A + B)^4 = 625$$

$$A + B = \pm 5 \rightarrow \text{equation 1}$$

Since $n = 4$, there are 5 terms, and the middle term is the 3rd term.

Hence,

$$r\text{th term} = \binom{n}{r-1} (a)^{n-r+1} (b)^{r+1}$$

$$n = 4$$

$$r = 3$$

$$a = Ax$$

$$b = By$$

$$3\text{rd term} = \binom{4}{3-1} (Ax)^{4-3+1} (By)^{3+1}$$

$$3\text{rd term} = \binom{4}{2} (Ax)^2 (By)^2$$

Equating the coefficients:

$$6A^2B^2 = 216$$

$$A^2B^2 = 36$$

$$AB = \pm 6$$

$$B = \pm \frac{6}{A} \rightarrow \text{equation 2}$$

Substituting equation 2 to equation 1:

$$A + \left(\pm \frac{6}{A}\right) = \pm 5$$

$$A^2 \pm 6 = \pm 5A$$

$$A^2 \mp 5A \pm 6 = 0$$

Case 1: $A^2 - 5A + 6 = 0$

$$A^2 - 5A + 6 = 0; A = 2; A = 3$$

Case 2: $A^2 + 5A + 6 = 0$

$$A^2 + 5A + 6 = 0; A = -2; A = -3$$

Case 3: $A^2 - 5A - 6 = 0$

$$A^2 - 5A - 6 = 0; A = -1; A = 6$$

Case 4: $A^2 + 5A - 6 = 0$

$$A^2 + 5A - 6 = 0; A = -6; A = 1$$

Therefore,

$$A = \{\pm 1, \pm 2, \pm 3, \pm 6\}$$

ALGEBRA: COMPLEX NUMBERS

1. If $A = 3 + i$ and $B = -2 - 3i$, find $A + B$.

A. $1 + 2i$	C. $-1 - 2i$
B. $1 - 2i$	D. $-1 + 2i$

2. If $A = 1 + 2i$ and $B = 2 - 3i$, find $A - B$.

A. $-1 + 5i$	C. $-1 - 2i$
c. $1 - 5i$	D. $-1 - 5i$

3. If $A = -2 - 5i$ and $B = 1 - 2i$, find $A \times B$.

A. $1 + 12i$	C. $12 + i$
B. $-12 + i$	D. $-12 - i$

4. If $A = 1 + 3i$ and $B = -1 + 3i$, find $\frac{A}{B}$.

A. $\frac{4}{5} + \frac{3}{5}i$	C. $\frac{3}{5} - \frac{4}{5}i$
B. $\frac{4}{5} - \frac{3}{5}i$	D. $\frac{3}{5} + \frac{4}{5}i$

5. If $A = 3 + 2i$, $B = 5 + 10i$ and $C = 4 - 9i$, find $\frac{A+B}{C} \times (B - C)$.

A. $-\frac{2356}{97} + \frac{1324}{97}i$	C. $\frac{2356}{97} - \frac{1234}{97}i$
B. $\frac{2356}{97} - \frac{1324}{97}i$	D. $-\frac{2356}{97} - \frac{1324}{97}i$

6. $i^{1072583}$ is also equal to ___?

A. i	C. -1
B. $-i$	D. 1

7. Evaluate $(i^{61} + i^{62} + i^{63} + i^{64})^{65}$.

A. 1	C. i
B. $-i$	D. 0

8. What is the polar form of $3 + 4i$?

A. $5 \angle 53.13^\circ$	C. $-5 \angle 36.87^\circ$
B. $-5 \angle 53.13^\circ$	D. $5 \angle 36.87^\circ$

9. Convert $-1 + 2i$ to exponential form.

A. $\sqrt{5}e^{2.034i}$	C. $\sqrt{5}e^{-2.034i}$
B. $\sqrt{5}e^{-0.464i}$	D. $\sqrt{5}e^{0.464i}$

10. Evaluate $(1 + i)^{16}$.

A. $-128 + 128i$	C. 256
c. 0	D. $256 - 256i$

11. Evaluate $(2 - 3i)^{92}$.

A. $-1.35 \times 10^{51} - 1.11 \times 10^{51}i$	C. $-1.35 \times 10^{51} + 1.11 \times 10^{51}i$
B. $1.11 \times 10^{51} - 1.35 \times 10^{51}i$	D. $-1.11 \times 10^{51} + 1.35 \times 10^{51}i$

12. Solve for the 3rd principal value of x in the equation: $x^3 = 64$.

A. $2 + 2\sqrt{3}i$	C. $-2 - 2\sqrt{3}i$
B. 4	D. $2 - 2\sqrt{3}i$

13. $(5 - 12i)^{3/4} = ?$ (third principal root)
 A. $4.35 - 5.29i$ C. $4.35 + 5.29i$
 B. $-5.29 - 4.35i$ D. $5.29 + 4.35i$
14. $i^i = ?$
 A. 4.8105 C. 0.2079
 B. $\sqrt{2}/2 + \sqrt{2}/2i$ D. $-\sqrt{2}/2 - \sqrt{2}/2i$
15. $(i^i)^i = ?$
 A. i C. $-i$
 B. 0.2079 D. 4.8105
16. $(2 - 3i)^i = ?$
 A. $0.7597 + 2.5616i$ C. $2.5616 + 0.7597i$
 B. $2.5616 - 0.7597i$ D. $-0.7597 + 2.5616i$
17. $(3 - 4i)^{(3+4i)} = ?$
 A. $-4442.53 + 2509.95i$ C. $4442.53 + 2509.95i$
 B. $4442.53 - 2509.95i$ D. $-4442.53 - 2509.95i$
18. $\ln i = ?$
 A. $\frac{\pi}{2}i$ C. 4.8105
 B. $\frac{\pi}{2}$ D. 0.2079
19. $\sqrt[i]{i} = ?$
 A. 4.8105 C. i
 B. $-i$ D. 0.2079
20. $\log_i(-5) = ?$
 A. $1 - 0.5122i$ C. $2 - 1.0246i$
 B. $1.0246 + 2i$ D. $\pi - 1.6094i$
21. $\sin i = ?$
 A. 1.5431 C. 1.1752i
 B. 1.5431i D. 1.1752
22. $\cos(1 + 2i) = ?$
 A. $2.0327 - 3.0519i$ C. 5.0846i
 B. $-1.0192i$ D. 5.0846
23. $\tan(3 - 2i) = ?$
 A. 1.2829i C. $-0.0099 - 0.9654i$
 B. 1.2829 D. -1.2829
24. One of the solutions of $\sin^{-1} i = ?$
 A. $\pi - \ln(1 + \sqrt{2})i$ C. $\ln(1 + \sqrt{2}) - \pi i$
 B. $\pi - \ln(1 - \sqrt{2})i$ D. $\ln(1 - \sqrt{2}) - \pi i$

25. $\cos^{-1}(3 + 4i) = ?$

A. $2.3055 - 0.9368i$

B. $0.9368 - 2.3055i$

C. $-2.3055 + 0.9368i$

D. $0.9368 + 2.3055i$

26. $\sinh(1 + i) = ?$

A. $0.6350 + 1.2985i$

B. $-0.6350 + 1.2985i$

C. $-0.6350 - 1.2985i$

D. $0.6350 - 1.2985i$

27. $\tanh^{-1}(-2 + 3i) = ?$

A. $0.1469 + 1.3390i$

B. $0.1469 - 1.3390i$

C. $-0.1469 - 1.3390i$

D. $-0.1469 + 1.3390i$

ALGEBRA: COMPLEX NUMBERS SOLUTIONS

1. If $A = 3 + i$ and $B = -2 - 3i$, find $A + B$.

Solution:

For operations involving complex numbers, go to MODE 2 (COMPLEX MODE)

Press the following keys:

$$A + B$$

Then press CALC.

Your calculator will ask you for the value of A and B, respectively:

$$A? A = 3 + i$$

$$B? B = -2 - 3i$$

Then press =

The answer will be **$1 - 2i$** .

2. If $A = 1 + 2i$ and $B = 2 - 3i$, find $A - B$.

Solution:

Press the following keys:

$$A - B$$

Then press CALC.

Your calculator will ask you for the value of A and B, respectively:

$$A? A = 1 + 2i$$

$$B? B = 2 - 3i$$

Then press =

The answer will be **$-1 + 5i$** .

3. If $A = -2 - 5i$ and $B = 1 - 2i$, find $A \times B$.

Solution:

Press the following keys:

$$A \times B$$

Then press CALC.

Your calculator will ask you for the value of A and B, respectively:

$$A? A = -2 - 5i$$

$$B? B = 1 - 2i$$

Then press =

The answer will be **$-12 - i$** .

4. If $A = 1 + 3i$ and $B = -1 + 3i$, find $\frac{A}{B}$.

Solution:

Press the following keys:

$$A \div B$$

Then press CALC.

Your calculator will ask you for the value of A and B, respectively:

$$A? A = 1 + 3i$$

$$B? B = -1 + 3i$$

Then press =

The answer will be **$\frac{4}{5} - \frac{3}{5}i$** .

5. If $A = 3 + 2i$, $B = 5 + 10i$ and $C = 4 - 9i$, find $\frac{A+B}{C} \times (B - C)$.

Solution:

Press the following keys:

$$\frac{A + B}{C} (B - C)$$

Then press CALC.

Your calculator will ask you for the value of A and B, respectively:

$$A? A = 3 + 2i$$

$$B? B = 5 + 10i$$

$$C? C = 4 - 9i$$

Then press =

$$\text{The answer will be } -\frac{2356}{97} - \frac{1324}{97}i = -24.29 - 13.65i.$$

6. $i^{1072583}$ is also equal to ___?

Solution:

Remember that:

$$i^1 = i$$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

Any exponent that is a multiple of 4 gives a value of 1.

Therefore,

$$i^{1072583} = i^{4(268145)+3} = i^{4(268145)} \cdot i^3 = (1)(i)^3 = -i$$

The answer will be $-i$.

7. Evaluate $(i^{61} + i^{62} + i^{63} + i^{64})^{65}$.

Solution:

$$(i^{61} + i^{62} + i^{63} + i^{64})^{65} = [i + (-1) + (-i) + 1]^{65} = 0^{65} = 0$$

The answer will be **0**.

8. What is the polar form of $3 + 4i$?

First Solution:

From the formula:

$$r \angle \theta = (a + bi); \text{ where } r = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1} \left(\frac{b}{a} \right) \text{ in degrees}$$

We have,

$$r = \sqrt{3^2 + 4^2} = 5$$

$$\theta = \tan^{-1} \left(\frac{4}{3} \right) = 53.13^\circ$$

Therefore,

$$3 + 4i = 5 \angle 53.13^\circ$$

Second Solution:

Type this in your calculator:

$$3 + 4i$$

And then press SHIFT-2-3 $\Rightarrow r \angle \theta$

The answer will be **5 \angle 53.13 $^\circ$**

9. Convert $-1 + 2i$ to exponential form.

Solution:

From the formula:

$$re^{i\theta} = (a + bi); \text{ where } r = \sqrt{a^2 + b^2}; \theta = \tan^{-1} \left(\frac{b}{a} \right) \text{ in radians}$$

We have:

$$r = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

$$\theta = \tan^{-1} \left(\frac{2}{-1} \right) = 116.57^\circ \cdot \frac{\pi}{180^\circ} = 2.034$$

Therefore,

$$-1 + 2i = \sqrt{5}e^{2.034i}$$

10. Evaluate $(1 + i)^{16}$.

First Solution:

Convert $1 + i$ into polar form:

$$1 + i = \sqrt{1^2 + 1^2} \angle \tan^{-1} \left(\frac{1}{1} \right) = \sqrt{2} \angle 45^\circ$$

Using Power of Complex Numbers:

$$(r \angle \theta)^n = r^n \angle n\theta$$

We have,

$$(\sqrt{2} \angle 45^\circ)^{16} = (\sqrt{2})^{16} \angle 16(45^\circ)$$

$$(\sqrt{2} \angle 45^\circ)^{16} = 256 \angle 2(360^\circ)$$

$$(\sqrt{2} \angle 45^\circ)^{16} = 256 \angle 0^\circ$$

$$(\sqrt{2} \angle 45^\circ)^{16} = \mathbf{256}$$

Second Solution:

By typing this in your calculator:

$$A^3 A^3 A^3 A^3 A^3 A$$

And then press CALC.

Note that your calculator does not recognize exponents greater than 3.

Your calculator will ask you for the value of A:

$$A? A = 1 + i$$

Then press =

The answer is **256**.

11. Evaluate $(2 - 3i)^{92}$.

Solution:

Change $2 - 3i$ to polar form:

$$\tan^{-1} \left(-\frac{3}{2} \right) = -56.31^\circ \rightarrow \text{STO } A$$

$$2 - 3i = \sqrt{2^2 + (-3)^2} \angle A$$

$$2 - 3i = \sqrt{13} \angle -56.31^\circ$$

$$(2 - 3i)^{92} = (\sqrt{13})^{92} \angle 92(-56.31^\circ)$$

$$(2 - 3i)^{92} = 13^{46} \angle (92A)$$

Change to rectangular form:

$$13^{46} \angle (92A)$$

Then press SHIFT 2-4 $\Rightarrow a + bi$

The answer will be $-1.35 \times 10^{51} - 1.11 \times 10^{51}i$.

12. Solve for the 3rd principal value of x in the equation: $x^3 = 64$.

Solution:

Using De Moivre's Formula:

$$[r \angle \theta]^{\frac{1}{n}} = r^{\frac{1}{n}} \angle \left(\frac{\theta + 360^\circ k}{n} \right); k = 0, 1, 2, 3, \dots, (n - 1)$$

We have,

$$(64 + 0i)^{\frac{1}{3}} = (64 \angle 0^\circ)^{\frac{1}{3}}$$

$$r = 64$$

$$n = 3$$

$$\theta = 0^\circ$$

When $k = 0$:

$$(64 + 0i)^{\frac{1}{3}} = 64^{\frac{1}{3}} \angle \frac{0^\circ + 360^\circ(0)}{3} = 4 \angle 0^\circ = 4$$

When $k = 1$:

$$(64 + 0i)^{\frac{1}{3}} = 64^{\frac{1}{3}} \angle \frac{0^\circ + 360^\circ(1)}{3} = 4 \angle 120^\circ = -2 + 2\sqrt{3}i$$

When $k = 2$:

$$(64 + 0i)^{\frac{1}{3}} = 64^{\frac{1}{3}} \angle \frac{0^\circ + 360^\circ(2)}{3} = 4 \angle 240^\circ = -2 - 2\sqrt{3}i$$

Therefore, the 3rd principal root is $-2 - 2\sqrt{3}i$.

13. $(5 - 12i)^{3/4} = ?$ (third principal root)

Solution:

$$(5 - 12i)^{3/4} = [(5 - 12i)^3]^{1/4}$$

$$(5 - 12i)^{3/4} = [(13 \angle -67.38^\circ)^3]^{1/4}$$

$$(5 - 12i)^{3/4} = [13^3 \angle 3(-67.38^\circ)]^{1/4}$$

$$(5 - 12i)^{3/4} = [2197 \angle -202.14^\circ]^{1/4}$$

$$(5 - 12i)^{3/4} = [2197 \angle 157.86^\circ]^{1/4}$$

When $k = 0$:

$$(5 - 12i)^{3/4} = 2197^{1/4} \angle \frac{157.86^\circ + 360^\circ(0)}{4} = 6.85 \angle 39.46^\circ = 5.29 + 4.35i$$

When $k = 1$:

$$\begin{aligned} (5 - 12i)^{3/4} &= 2197^{1/4} \angle \frac{-202.14^\circ + 360^\circ(1)}{4} = 6.85 \angle 129.46^\circ \\ &= -4.35 + 5.29i \end{aligned}$$

When $k = 2$:

$$\begin{aligned}(5 - 12i)^{3/4} &= 2197^{1/4} \angle \frac{-202.14^\circ + 360^\circ(2)}{4} = 6.85 \angle -140.54^\circ \\ &= -5.29 - 4.35i\end{aligned}$$

When $k = 3$:

$$\begin{aligned}(5 - 12i)^{3/4} &= 2197^{1/4} \angle \frac{-202.14^\circ + 360^\circ(3)}{4} = 6.85 \angle -50.54^\circ \\ &= 4.35 - 5.29i\end{aligned}$$

Therefore, the 3rd principal root is **-5.29 - 4.35i**.

14. $i^i = ?$

Solution:

$$\begin{aligned}i^i &= (0 + i)^i \\ i^i &= (1 \angle 90^\circ)^i \\ i^i &= (e^{\pi/2i})^i \\ i^i &= e^{-\pi/2} \\ i^i &= \mathbf{0.2079}\end{aligned}$$

15. $(i^i)^i = ?$

Solution:

$$\begin{aligned}(i^i)^i &= [(0 + i)^i]^i \\ (i^i)^i &= [(1 \angle 90^\circ)^i]^i \\ (i^i)^i &= [(e^{\pi/2i})^i]^i \\ (i^i)^i &= (e^{-\pi/2})^i \\ (i^i)^i &= (e^{-\pi/2i}) \\ (i^i)^i &= \mathbf{-i}\end{aligned}$$

16. $(2 - 3i)^i = ?$

Solution:

$$\begin{aligned}(2 - 3i)^i &= (\sqrt{13} \angle -56.31^\circ)^i \\ (2 - 3i)^i &= (\sqrt{13} e^{-0.9828i})^i \\ (2 - 3i)^i &= (e^{\ln \sqrt{13} - 0.9828i})^i \\ (2 - 3i)^i &= e^{0.9828 + \ln \sqrt{13}i} \\ (2 - 3i)^i &= e^{0.9828} \cdot e^{\ln \sqrt{13}i} \\ (2 - 3i)^i &= 2.6719 \cdot e^{1.2825i} \\ (2 - 3i)^i &= 2.6719 \angle 73.48^\circ \\ (2 - 3i)^i &= \mathbf{0.7597 + 2.5616i}\end{aligned}$$

17. $(3 - 4i)^{(3+4i)} = ?$

Solution:

$$\begin{aligned} (3 - 4i)^{(3+4i)} &= (5\angle - 53.13^\circ)^{(3+4i)} \\ (3 - 4i)^{(3+4i)} &= (5e^{-0.9273i})^{(3+4i)} \\ (3 - 4i)^{(3+4i)} &= (e^{\ln 5 - 0.9273i})^{(3+4i)} \\ (3 - 4i)^{(3+4i)} &= e^{(\ln 5 - 0.9273i)(3+4i)} \\ (3 - 4i)^{(3+4i)} &= e^{8.5374 + 3.6559i} \\ (3 - 4i)^{(3+4i)} &= e^{8.5374} \cdot e^{3.6559i} \\ (3 - 4i)^{(3+4i)} &= 5102.54\angle 209.47^\circ \\ (3 - 4i)^{(3+4i)} &= \mathbf{4442.53 - 2509.95i} \end{aligned}$$

18. $\ln i = ?$

Solution:

$$\begin{aligned} \ln i &= \ln(0 + i) \\ \ln i &= \ln\left(e^{\frac{\pi}{2}i}\right) \\ \ln i &= \frac{\pi}{2}i \end{aligned}$$

19. $\sqrt[i]{i} = ?$

Solution:

$$\begin{aligned} \sqrt[i]{i} &= i^{1/i} \\ \sqrt[i]{i} &= (0 + i)^{1/i} \\ \sqrt[i]{i} &= (e^{\pi/2i})^{1/i} \\ \sqrt[i]{i} &= e^{\pi/2} \\ \sqrt[i]{i} &= \mathbf{4.8105} \end{aligned}$$

20. $\log_i(-5) = ?$

Solution:

$$\begin{aligned} \log_i(-5) &= \frac{\log(-5)}{\log i} \\ \log_i(-5) &= \frac{\ln(-5)}{\ln i} \\ \log_i(-5) &= \frac{\ln(-5 + 0i)}{\ln(0 + i)} \\ \log_i(-5) &= \frac{\ln(5e^{\pi i})}{\ln\left(e^{\frac{\pi}{2}i}\right)} \\ \log_i(-5) &= \frac{\ln(e^{\ln 5 + \pi i})}{\ln\left(e^{\frac{\pi}{2}i}\right)} \\ \log_i(-5) &= \frac{\ln 5 + \pi i}{\frac{\pi}{2}i} \\ \log_i(-5) &= \mathbf{2 - 1.0246i} \end{aligned}$$

21. $\sin i = ?$

Solution:

Note:

By Euler's Formula:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{-i\theta} = \cos -\theta + i \sin -\theta = \cos \theta - i \sin \theta$$

Let $\theta = ix$:

$$e^{-x} = \cos ix + i \sin ix$$

$$e^x = \cos ix - i \sin ix$$

Adding these two equations:

$$e^x + e^{-x} = 2 \cos ix$$

$$\cos ix = \frac{e^x + e^{-x}}{2}$$

$$\cos ix = \cosh x$$

Subtract the second equation to the first equation, we have:

$$e^x - e^{-x} = -2i \sin ix$$

$$\sin ix = \frac{e^x - e^{-x}}{-2i}$$

$$\sin ix = i \cdot \frac{e^x - e^{-x}}{2}$$

$$\sin ix = i \sinh x$$

When $x = 1$:

$$\sin i = i \sinh 1$$

$$\sin i = \mathbf{1.1752i}$$

22. $\cos(1 + 2i) = ?$

Solution:

$$\cos(1 + 2i) = \cos(1) \cos(2i) - \sin(1) \sin(2i)$$

$$\cos(1 + 2i) = \cos(1) \cosh(2) - \sin(1) i \sinh(2)$$

Note: Put your calculator into radian mode:

$$\cos(1 + 2i) = \mathbf{2.0327 - 3.0519i}$$

23. $\tan(3 - 2i) = ?$

Solution:

$$\tan(3 - 2i) = \frac{\sin(3 - 2i)}{\cos(3 - 2i)}$$

$$\tan(3 - 2i) = \frac{\sin(3) \cos(2i) - \cos(3) \sin(2i)}{\cos(3) \cos(2i) + \sin(3) \sin(2i)}$$

$$\tan(3 - 2i) = \frac{\sin(3) \cosh(2) - \cos(3) i \sinh(2)}{\cos(3) \cosh(2) + \sin(3) i \sinh(2)}$$

Note: Put your calculator into radian mode:

$$\tan(3 - 2i) = \mathbf{-0.0099 - 0.9654i}$$

24. One of the solutions of $\sin^{-1} i = ?$

Solution:

Let:

$$y = \sin^{-1} i$$

Therefore,

$$\sin y = i$$

Note:

$$\sin ix = i \sinh x$$

Let:

$$x = \frac{y}{i}$$

We have,

$$\begin{aligned} \sin\left(i \cdot \frac{y}{i}\right) &= i \sinh\left(\frac{y}{i}\right) \\ \sin(y) &= i \sinh(-iy) \end{aligned}$$

But,

$$\sin y = i$$

Therefore,

$$\begin{aligned} i \sinh(-iy) &= i \\ \sinh(-iy) &= 1 \end{aligned}$$

Applying identities of hyperbolic functions:

$$-\sinh y = \sinh(-y)$$

Hence,

$$\begin{aligned} -\sinh(iy) &= 1 \\ \sinh(iy) &= -1 \\ \frac{e^{iy} - e^{-iy}}{2} &= -1 \\ e^{iy} - e^{-iy} &= -2 \end{aligned}$$

Multiply both sides by e^{iy} :

$$\begin{aligned} e^{2iy} - 1 &= -2e^{iy} \\ e^{2iy} + 2e^{iy} - 1 &= 0 \end{aligned}$$

Go to MODE 5-3 (3: $aX^2 + bX + c = 0$)

$$\begin{aligned} a &= 1 \\ b &= 2 \\ c &= -1 \end{aligned}$$

Solving for the roots:

$$e^{iy} = -1 \pm \sqrt{2}$$

Since there are two roots, there are also two values of y :

For the first root:

$$e^{iy} = -1 + \sqrt{2}$$

Get the natural logarithm of both sides:

$$\begin{aligned} \ln(e^{iy}) &= \ln(-1 + \sqrt{2}) \\ iy &= -0.8814 \\ y &= 0.8814i \end{aligned}$$

For the second root:

$$\begin{aligned}
 e^{iy} &= \ln(-1 - \sqrt{2}) \\
 e^{iy} &= \ln(-2.4142) \\
 e^{iy} &= \ln(2.4142 \angle 180^\circ) \\
 e^{iy} &= \ln(2.4142e^{\pi i}) \\
 e^{iy} &= \ln(e^{\ln 2.4142 + \pi i}) \\
 e^{iy} &= e^{\ln 2.4142 + \pi i} \\
 iy &= \ln 2.4142 + \pi i \\
 y &= \pi - 0.8814i \\
 y &= \mathbf{3.1416 - 0.8814i}
 \end{aligned}$$

25. $\cos^{-1}(3 + 4i) = ?$

Solution:

$$\cos ix = \cosh x$$

Let:

$$y = \cos^{-1}(3 + 4i)$$

Therefore,

$$3 + 4i = \cos y$$

By letting $x = \frac{y}{i}$:

$$\begin{aligned}
 \cos\left(i \cdot \frac{y}{i}\right) &= \cosh \frac{y}{i} \\
 \cos y &= \cosh(-iy) \\
 \cos y &= \cosh iy
 \end{aligned}$$

So, we have,

$$\begin{aligned}
 \cosh iy &= 3 + 4i \\
 \frac{e^{iy} + e^{-iy}}{2} &= 3 + 4i \\
 e^{iy} + e^{-iy} &= 6 + 8i
 \end{aligned}$$

Multiply both sides by e^{iy} :

$$\begin{aligned}
 e^{2iy} + 1 &= (6 + 8i)e^{iy} \\
 e^{2iy} - (6 + 8i)e^{iy} + 1 &= 0
 \end{aligned}$$

Using quadratic formula:

$$\begin{aligned}
 e^{iy} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
 a &= 1 \\
 b &= -(6 + 8i) \\
 c &= 1
 \end{aligned}$$

We have:

$$e^{iy} = \frac{-[-(6 + 8i)] \pm \sqrt{[-(6 + 8i)]^2 - 4(1)(1)}}{2(1)}$$

$$e^{iy} = \frac{(6 + 8i) \pm \sqrt{-32 + 96i}}{2}$$

$$e^{iy} = \frac{(6 + 8i) \pm (32\sqrt{10} \angle 108.43^\circ)^{1/2}}{2}$$

$$e^{iy} = \frac{(6 + 8i) \pm (32\sqrt{10}e^{1.8925i})^{1/2}}{2}$$

$$e^{iy} = \frac{(6 + 8i) \pm (5.8819 + 8.1607i)}{2}$$

$$e^{iy} = (3 + 4i) \pm (2.9409 + 4.0803i)$$

Then store $2.9409 + 4.0803i$ to A.

For the first root:

$$e^{iy} = (3 + 4i) + (2.9409 + 4.0803i)$$

$$e^{iy} = 5.9409 + 8.0803i$$

$$e^{iy} = 10.0293 \angle 53.68^\circ$$

$$e^{iy} = 10.0293e^{0.9368i}$$

$$\ln e^{iy} = \ln(e^{\ln 10.0293 + 0.9368i})$$

$$iy = \ln 10.0293 + 0.9368i$$

$$y = \mathbf{0.9368 - 2.3055i}$$

For the second root:

$$e^{iy} = (3 + 4i) - (2.9409 + 4.0803i)$$

$$e^{iy} = 0.0591 - 0.0803i$$

$$e^{iy} = 0.0997 \angle -53.68^\circ$$

$$e^{iy} = 0.0997e^{-0.9368i}$$

$$\ln e^{iy} = \ln(e^{\ln 0.0997 - 0.9368i})$$

$$iy = \ln 0.0997 - 0.9368i$$

$$y = -0.9368 + 2.3055i$$

26. $\sinh(1 + i) = ?$

Solution:

$$\sinh(1 + i) = \frac{e^{(1+i)} - e^{-(1+i)}}{2}$$

$$\sinh(1 + i) = \frac{e^{(1+i)} - e^{(-1-i)}}{2}$$

$$\sinh(1 + i) = \frac{e^1 \cdot e^i - e^{-1} \cdot e^{-i}}{2}$$

$$\sinh(1 + i) = \frac{e \angle 57.30^\circ - \frac{1}{e} \angle -57.30^\circ}{2}$$

$$\sinh(1 + i) = \mathbf{0.6350 + 1.2985i}$$

27. $\tanh^{-1}(-2 + 3i) = ?$

Solution:

$$\begin{aligned}
 y &= \tanh^{-1}(-2 + 3i) \\
 \tanh y &= -2 + 3i \\
 \frac{\sinh y}{\cosh y} &= -2 + 3i \\
 \frac{\left(\frac{e^y - e^{-y}}{2}\right)}{\left(\frac{e^y + e^{-y}}{2}\right)} &= -2 + 3i \\
 \frac{e^y - e^{-y}}{e^y + e^{-y}} &= -2 + 3i \\
 e^y - e^{-y} &= (-2 + 3i)(e^y + e^{-y}) \\
 e^y - (-2 + 3i)e^y - e^{-y} - (-2 + 3i)e^{-y} &= 0 \\
 (3 - 3i)e^y + (1 - 3i)e^{-y} &= 0
 \end{aligned}$$

Multiply both sides by e^y :

$$\begin{aligned}
 (3 - 3i)e^{2y} + (1 - 3i) &= 0 \\
 e^{2y} + \left(\frac{2}{3} - \frac{1}{3}i\right) &= 0 \\
 e^{2y} &= -\frac{2}{3} + \frac{1}{3}i \\
 e^{2y} &= \frac{\sqrt{5}}{3} \angle 153.43^\circ \\
 e^{2y} &= \frac{\sqrt{5}}{3} e^{2.6779i} \\
 e^{2y} &= e^{\ln\left(\frac{\sqrt{5}}{3}\right) + 2.6779i} \\
 2y &= \ln \frac{\sqrt{5}}{3} + 2.6779i \\
 y &= \frac{1}{2} \ln \frac{\sqrt{5}}{3} + 1.3390i \\
 y &= -\mathbf{0.1469} + \mathbf{1.3390i}
 \end{aligned}$$

MATRICES AND DETERMINANTS

1. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$, and $B = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$, find $A + B$.

A. $\begin{bmatrix} 3 & 5 & 3 \\ -1 & 3 & 9 \end{bmatrix}$

C. $\begin{bmatrix} 3 & -1 \\ 5 & 3 \\ 3 & 9 \end{bmatrix}$

B. $\begin{bmatrix} -1 & -1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$

D. $\begin{bmatrix} -1 & 1 \\ -1 & -1 \\ 3 & -1 \end{bmatrix}$

2. If $A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$, find $3A$.

A. $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$

C. $\begin{bmatrix} 1 & -8 \\ 8 & 27 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$

D. $\begin{bmatrix} 3 & -6 \\ 6 & 9 \end{bmatrix}$

3. If $A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$, find A^2 .

A. $\begin{bmatrix} 4 & 1 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 4 & 1 & 1 \\ 0 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 5 & -3 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{bmatrix}$

D. $\begin{bmatrix} 5 & 2 & 3 \\ 1 & 4 & 2 \\ -3 & 1 & -1 \end{bmatrix}$

4. Calculate the product $C = AB$ of the matrices A and B .

$$A = \begin{bmatrix} 1 & 8 \\ 4 & -2 \\ -5 & 3 \end{bmatrix}, B = \begin{bmatrix} 6 & -7 \\ -1 & 2 \end{bmatrix}$$

A. $\begin{bmatrix} -2 & 9 \\ 26 & -32 \\ -33 & 41 \end{bmatrix}$

C. $\begin{bmatrix} 14 & 9 \\ 22 & -32 \\ -27 & 41 \end{bmatrix}$

B. $\begin{bmatrix} -2 & 9 \\ 26 & -32 \\ -27 & 29 \end{bmatrix}$

D. cannot be determined

5. Which of the following is a symmetric matrix?

A. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

C. $\begin{bmatrix} 6 & 12 & 0 \\ 15 & -9 & -37 \\ 40 & 0 & 11 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

D. $\begin{bmatrix} 6 & 15 & -24 \\ 15 & -9 & 13 \\ -24 & 13 & 8 \end{bmatrix}$

6. Find the transpose of the matrix given below.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

A. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

C. $\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

D. cannot be determined

7. Find $(AB)^T$ if

$$A = \begin{bmatrix} 9 & -5 \\ 2 & 1 \\ -3 & 4 \end{bmatrix}, B = \begin{bmatrix} 6 & -1 & 10 \\ -2 & 7 & 5 \end{bmatrix}$$

A. $\begin{bmatrix} 22 & -19 \\ 9 & 37 \end{bmatrix}$

B. $\begin{bmatrix} 22 & 9 \\ -19 & 37 \end{bmatrix}$

C. $\begin{bmatrix} 64 & 10 & -26 \\ -44 & 5 & 31 \\ 65 & 25 & -10 \end{bmatrix}$

D. $\begin{bmatrix} 64 & -44 & 65 \\ 10 & 5 & 25 \\ -26 & 31 & -10 \end{bmatrix}$

8. Find the inverse of the matrix given below.

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

A. $\begin{bmatrix} -2 & 1 \\ -1.5 & 0.5 \end{bmatrix}$

B. $\begin{bmatrix} 2 & -1 \\ 1.5 & -0.5 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 3 \\ -2 & -4 \end{bmatrix}$

D. $\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$

9. For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$, find the determinant of matrix A.

A. 121

B. -121

C. 0

D. -107

10. Find the minor of 6 in the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

A. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$

C. $\begin{bmatrix} -1 & -2 \\ -7 & -8 \end{bmatrix}$

D. $\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$

11. Find the cofactor of 6 in the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

A. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$

C. $\begin{bmatrix} -1 & -2 \\ -7 & -8 \end{bmatrix}$

D. $\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$

12. Solve for the value of determinant D of the given matrix:

$$A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 6 & 5 & 2 & 9 \\ 3 & 9 & 7 & 3 \\ 4 & 6 & 7 & 2 \end{bmatrix}$$

A. 121

B. -121

C. 0

D. -813

13. For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$, find the determinant of the adjoint of matrix

- A.
- A. 49
- B. -49
- C. 1/49
- D. -1/49

14. Suppose A is a square matrix of order 3. If the determinant of A is equal to -2 , find the determinant of $A^2(2A)^T$.

- A. -16
- B. -32
- C. -64
- D. -128

15. Which of the following relations describes the property of an orthogonal matrix?

- A. $A^{-1} = A^T$
- B. $-A^{-1} = A^T$
- C. $A^{-1} = (A^T)^{-1}$
- D. $A = A^{-1}$

MATRICES AND DETERMINANTS SOLUTIONS

1. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$, and $B = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$, find $A + B$.

Solution:

To evaluate $A + B$, you just add the corresponding elements.

$$A + B = \begin{bmatrix} 1 + 2 & 2 + 3 & 3 + 0 \\ 0 + (-1) & 1 + 2 & 4 + 5 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ -1 & 3 & 9 \end{bmatrix}$$

2. If $A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$, find $3A$.

Solution:

To evaluate $3A$, multiply each element of the matrix by the constant (in this example, $c = 3$).

$$3A = \begin{bmatrix} 3(1) & 3(-2) \\ 3(2) & 3(3) \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 6 & 9 \end{bmatrix}$$

3. If $A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$, find A^2 .

Solution:

Note: The multiplication of two matrices can be carried out only if the number of columns of the first matrix equals the number of rows of the second matrix. Such matrices are referred to as being conformable for multiplication.

$$A^2 = A \cdot A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

When two conformable matrices are multiplied, the product matrix thus obtained will have the number of rows of the first matrix and the number of columns of the second matrix. Thus, if a matrix A of order $m \times n$ is post multiplied by a matrix B of order $n \times s$, then the product matrix C will be of order $m \times s$:

$$A^2 = A \cdot A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2(2) + (-1)(0) + 1(1) & 2(-1) + (-1)(1) + 1(0) & 2(1) + (-1)(2) + 1(1) \\ 0(2) + 1(0) + 2(1) & 0(-1) + 1(1) + 2(0) & 0(1) + 1(2) + 2(1) \\ 1(2) + 0(0) + 1(1) & 1(-1) + 0(1) + 1(0) & 1(1) + 0(2) + 1(1) \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 5 & -3 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{bmatrix}$$

4. Calculate the product $C = AB$ of the matrices A and B .

$$A = \begin{bmatrix} 1 & 8 \\ 4 & -2 \\ -5 & 3 \end{bmatrix}, B = \begin{bmatrix} 6 & -7 \\ -1 & 2 \end{bmatrix}$$

Solution:

$$C = \begin{bmatrix} 1 & 8 \\ 4 & -2 \\ -5 & 3 \end{bmatrix} \cdot \begin{bmatrix} 6 & -7 \\ -1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1(6) + 8(-1) & 1(-7) + 8(2) \\ 4(6) + (-2)(-1) & 4(-7) + (-2)(2) \\ -5(6) + 3(-1) & -5(-7) + 3(2) \end{bmatrix} = \begin{bmatrix} -2 & 9 \\ 26 & -32 \\ -33 & 41 \end{bmatrix}$$

5. Which of the following is a symmetric matrix?

From the given choices:

A matrix with all its elements arranged in a single column is called a column matrix. The matrix described in a is an example of a column matrix.

A matrix with all its elements arranged in a single row is called a row matrix. The matrix described in b is an example of a row matrix.

If a matrix has the same number of rows and columns (i.e., $m = n$), it is called a square matrix. Letter c is an example of a square matrix.

When the elements of a square matrix are symmetric about its main diagonal (i.e., $A_{ij} = A_{ji}$), it is termed as a symmetric matrix. Letter d is an example of a symmetric matrix.

6. Find the transpose of the matrix given below.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

First Solution:

The transpose of a matrix is obtained by interchanging its corresponding rows and columns. The transposed matrix is commonly identified by placing a subscript T on the symbol of the original matrix.

Let the given matrix as matrix A:

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Second solution:

Using your fx-991ES PLUS,

Use the Matrix Mode:

Mode – 6 – 1 – 4 (Mat A 2×3)

Then input the corresponding matrix.

To get the transpose of the matrix, just use the syntax:

Shift – 4 – 8 – Shift – 4 – 3

$$\text{Trn(Mat A)} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Which yields to the same answer.

7. Find $(AB)^T$ if

$$A = \begin{bmatrix} 9 & -5 \\ 2 & 1 \\ -3 & 4 \end{bmatrix}, B = \begin{bmatrix} 6 & -1 & 10 \\ -2 & 7 & 5 \end{bmatrix}$$

First Solution:

Another useful property of a matrix transposition is that the transpose of a product of matrices equals the product of the transposed matrices in reverse order. Thus,

$$(AB)^T = B^T A^T$$

$$AB = \begin{bmatrix} 9 & -5 \\ 2 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 6 & -1 & 10 \\ -2 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 64 & -44 & 65 \\ 10 & 5 & 25 \\ -26 & 31 & -10 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 64 & 10 & -26 \\ -44 & 5 & 31 \\ 65 & 25 & -10 \end{bmatrix}$$

Or:

$$(AB)^T = B^T A^T = \begin{bmatrix} 6 & -2 \\ -1 & 7 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} 6 & 2 & -3 \\ -5 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 64 & 10 & -26 \\ -44 & 5 & 31 \\ 65 & 25 & -10 \end{bmatrix}$$

Second solution:

Using your fx-991ES PLUS:

Using the MATRIX MODE:

Mode – 6 – 1 – 2 (Mat A 3×2)

Then input the corresponding matrix.

To input matrix B, use the syntax:

Shift – 4 – 1 – 2 – 4 (Dim – Mat B – 2×3)

Then input the corresponding matrix.

To get the transpose of the product of matrices, just use the syntax:

Shift – 4 – 8 – Shift – 4 – 3 – Shift – 4 – 4

$$\text{Trn(MatAMatB)} = \begin{bmatrix} 64 & 10 & -26 \\ -44 & 5 & 31 \\ 65 & 25 & -10 \end{bmatrix}$$

Which yields to the same answer.

8. Find the inverse of the matrix given below.

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

Solution:

If A and B are square matrices such that $AB = BA = I$, then B is called the inverse of A and we write $B = A^{-1}$ (B equals A inverse). The matrix B also has A as its inverse and we write $A = B^{-1}$.

Note: Not every square matrix has an inverse. However, that if A has an inverse then that inverse is unique.

$$A \cdot A^{-1} = I$$

$$\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$1(a) + (-2)(c) = 1 \rightarrow (1)$$

$$1(b) + (-2)(d) = 0 \rightarrow (2)$$

$$3(a) + (-4)(c) = 0 \rightarrow (3)$$

$$3(b) + (-4)(d) = 1 \rightarrow (4)$$

Solving equations (1) and (3); and (2) and (4), we have

$$a = -2$$

$$b = 1$$

$$c = -1.5$$

$$d = 0.5$$

Therefore,

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ -1.5 & 0.5 \end{bmatrix}$$

Second solution:

Using your fx-991ES PLUS:

Using the MATRIX MODE:

Mode – 6 – 1 – 5 (Mat A 2×2)

Then input the corresponding matrix.

To get the inverse of the matrix, just use the syntax:

Shift – 4 – 3 – x^{-1}

$$\text{Mat } A^{-1} = \begin{bmatrix} -2 & 1 \\ -1.5 & 0.5 \end{bmatrix}$$

Which yields to the same answer.

9. For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$, find the determinant of matrix A.

Solution:

The determinant of a matrix is a number associated with a square matrix.

There are many ways to solve for the determinant of a square matrix. One example is the diagonal multiplication (this technique is only applicable for a 2 x 2 and 3 x 3 matrix).

$$\det(A) = |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 4 & -5 & 2 & 4 \\ 3 & -5 & 6 & 3 & -5 \end{vmatrix}$$

$$|A| = [1(4)(6) + 2(-5)(3) + 3(2)(-5)] - [3(4)(3) + (-5)(-5)(1) + 6(2)(2)]$$

$$|A| = -121$$

Or by the use of your calculator,

Input the given matrix to matrix A:

To get the determinant, follow this syntax:

Shift – 4 – 7 – Shift – 4 – 3

$$\det(\text{MatA}) = -121$$

10. Find the minor of 6 in the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Solution:

A minor is defined as a value computed from the determinant of a square matrix which is obtained after crossing out a row and a column corresponding to the element that is under consideration. It is denoted by M_{ij} .

Thus, the minor of 6 in the given matrix is:

$$M_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

11. Find the cofactor of 6 in the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Solution:

The cofactor is defined as the signed minor. It is obtained by multiplying M_{ij} to $(-1)^{i+j}$, or simply $(-1)^{i+j}M_{ij}$. It is denoted as α_{ij} .

Thus, the cofactor of 6 in the given matrix is:

$$\alpha_{23} = (-1)^{(2+3)} \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} = (-1)^5 \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -7 & -8 \end{bmatrix}$$

12. Solve for the value of determinant D of the given matrix:

$$A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 6 & 5 & 2 & 9 \\ 3 & 9 & 7 & 3 \\ 4 & 6 & 7 & 2 \end{bmatrix}$$

Solution:

One way to solve the determinant of a square matrix is by cofactor method.

$$\begin{aligned} |D| &= 1 \begin{vmatrix} 5 & 2 & 9 \\ 9 & 7 & 3 \\ 6 & 7 & 2 \end{vmatrix} - 2 \begin{vmatrix} 6 & 2 & 9 \\ 3 & 7 & 3 \\ 4 & 7 & 2 \end{vmatrix} + 4 \begin{vmatrix} 6 & 5 & 9 \\ 3 & 9 & 3 \\ 4 & 6 & 2 \end{vmatrix} - 5 \begin{vmatrix} 6 & 5 & 2 \\ 3 & 9 & 7 \\ 4 & 6 & 7 \end{vmatrix} \\ |D| &= 1(154) - 2(-93) + 4(-132) - 5(125) = -813 \end{aligned}$$

13. For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$, find the determinant of the adjoint of matrix

A.

First Solution:

The adjoint of a matrix is defined as the transpose of the determinant of the cofactors of each element. The adjoint of matrix A can be denoted as A_{ADJ} or $\text{adj } A$.

$$\begin{aligned} \text{adj } A &= \begin{bmatrix} \begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix} & -\begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix} \\ -\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} & -\begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} \\ \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} & -\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \end{bmatrix}^T \\ \text{adj } A &= \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}^T \\ \text{adj } A &= \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix} \end{aligned}$$

Therefore:

$$|\text{adj } A| = 49$$

Second Solution:

Using the relationship between the inverse and adjoint:

$$A^{-1} = \frac{\text{adj } A}{|A|} \rightarrow \text{adj } A = |A| \cdot A^{-1}$$

Using Matrix Mode:

Input the given matrix to Matrix A:

Mode – 6 – 1 – 1 (MatA 3 x 3)

After which, follow the syntax:

Shift – 4 – 7 – Shift – 4 – 3 –) – × – Shift – 4 – 3 – x⁻¹

$$\det(\text{MatA}) \times \text{MatA}^{-1} = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

To get the determinant, use the syntax:

Shift – 4 – 7 – Shift – 4 – 6 –)

$$\det(\text{MatAns}) = \mathbf{49}$$

14. Suppose A is a square matrix of order 3. If the determinant of A is equal to -2, find the determinant of A²(2A)^T.

Solution:

Think of a 3 x 3 matrix that has a determinant equal to -2. The best way is to form a diagonal matrix.

For example, let matrix A as:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Note: A square matrix with all of its off-diagonal elements equal to zero (i.e., A_{ij} = 0 for i ≠ j), is called a **diagonal matrix**.

Then input this matrix to Matrix A of your calculator.

After which, follow this syntax to obtain what is required:

Shift – 4 – 7 – Shift – 4 – 3 – x² – × – Shift – 4 – 8 – 2 – Shift – 4 – 3 –) –)

$$\det(\text{MatA}^2 \times \text{Trn}(2\text{MatA})) = \mathbf{-64}$$

15. Which of the following relations describes the property of an orthogonal matrix?

Solution: If the inverse of a matrix is equal to its transpose, the matrix is referred to as an orthogonal matrix. In other words, a matrix **A** is orthogonal if:

$$\mathbf{A^{-1} = A^T}$$

PLANE AND SPHERICAL TRIGONOMETRY

Situation:

A nautical mile is the length of an arc, on Earth's equator, that subtends a central angle. The equatorial radius of Earth is about 3960 statute miles.

- Convert 1 nautical mile to statute miles.

A. 0.932	C. 0.868
B. 1.073	D. 1.152
- Determine what percent of Earth's circumference is covered by a trip from Los Angeles, California, to Honolulu, Hawaii (2217 nautical miles).

A. 55.988%	C. 60.08%
B. 10.26%	D. 11.25%
- Which of the following is the value of x ? $\frac{1+\sin 2\theta-\cos 2\theta}{1+\sin 2\theta+\cos 2\theta} = x$

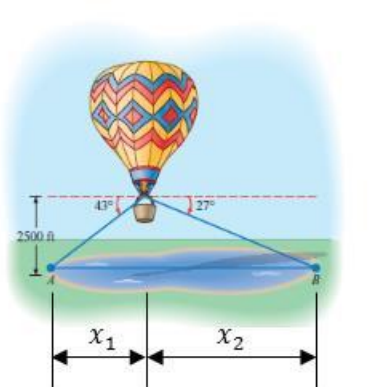
A. $2 \sin \theta$	C. $3 \cos \theta + 2$
B. $\tan \theta$	D. $\sin^2 \theta$

- If $\tan \theta = \frac{4}{3}$ and θ is an angle in the third quadrant, evaluate:

$$\frac{\sin(180 + \theta)}{\sec(270 + \theta)} + \frac{\cos(360 - \theta)}{\csc(90 + \theta)}$$

- | | |
|---------|----------|
| A. 0.28 | C. -0.28 |
| B. 3.57 | D. 1.00 |

- The angle of depression to one side of a lake, measured from a balloon 2500 feet above the lake as shown in the accompanying figure, is 43° . The angle of depression to the opposite side of the lake is 27° . Find the width of the lake.



- | | |
|-----------------|-----------------|
| A. 3605.101 ft. | C. 7237.814 ft. |
| B. 3954.735 ft. | D. 7587.778 ft. |

- From a point A on a line from the base of the Washington Monument, the angle of elevation to the top of the monument is 42° . From a point 100 feet away from A and on the same line, the angle to the top is 37.77° . Find the height of the Washington Monument.

A. 555.634 ft.	C. 617.094 ft.
B. 685.353 ft.	D. 645.675 ft.

Situation:

From point A, at street level and 205 feet from the base of a building, the angle of elevation to the top of the building is 23.1° . Also, from point A the angle of elevation to the top of a neon sign, which is at the top the building, is 25.9° .

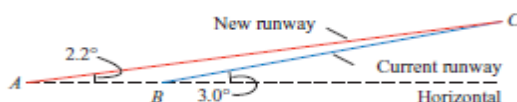
7. Determine the height of the building.

A. 480.616 ft.	C. 87.440 ft.
B. 79.923 ft.	D. 525.798 ft.
8. How tall is the sign?

A. 381.073 ft.	C. 12.103 ft.
B. 19.620 ft.	D. 426.255 ft.
9. A right triangle ACB with the right angle at C has legs 5 m and 12 m. Find the length of a line drawn from C perpendicular to the hypotenuse.

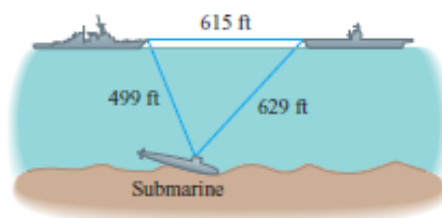
A. 4.615 m	C. 2.308 m
B. 4.156 m	D. 3.145 m
10. The bases of a parcel of land in the form of a trapezoid are 92.6 m and 75.8 m, respectively. The angle at the extremities of the longer base is 72° and 43° , respectively. Find the perimeter of the parcel of land.

A. 198.672 m	C. 183.817 m
B. 250.123 m	D. 150.555 m
11. An airport runway is 3550 feet long and has an incline of 3.0° . The airport planning committee plans to replace this runway with a new runway, as shown in the following figure. The new runway will be inclined at an angle of 2.2° . What will be the length of the new runway?



- | | |
|-----------------|-----------------|
| A. 3554.880 ft. | C. 4839.887 ft. |
| B. 3552.619 ft. | D. 2600.315 ft. |
12. Two observers, in the same vertical plane as a kite and 30 feet apart, observe the kite at angles of 72° and 78° , respectively. Find the height of the kite.

A. 36.514 ft.	C. 42.315 ft.
B. 65.221 ft.	D. 55.816 ft.
 13. Use the distances shown in the following figure to determine the depth of the submarine below the surface of the water. Assume that the line segment between the surface ships is directly above the submarine.



- | | |
|----------------|----------------|
| A. 578.793 ft. | C. 462.116 ft. |
| B. 582.507 ft. | D. 188.280 ft. |
14. A right triangle with sides 4.32 and 2.41 inches long respectively is inscribed in a circle. What is the diameter of the circle?

A. 2.473 ft.	C. 4.485 ft.
B. 4.947 ft.	D. 3.236 ft.

Situation:

An aero plane flies at N 26°35' W for 137.2 miles, then S 53°25' W for 62.4 miles.

15. How far must it go?

- | | |
|------------------|------------------|
| A. 140.514 miles | C. 189.465 miles |
| B. 120.517 miles | D. 105.875 miles |

16. In what direction should it then fly to return to the starting point in a straight line?

- | | |
|---------------------|---------------------|
| A. N 52°21'02.5" W | C. S 37°38'57.95" E |
| B. N 37°38'57.95" W | D. S 52°31'02.5" E |

17. Two straight roads intersect to form an angle of 75°. Find the shortest distance from one road to a gas station on the other road that is 1000 m from the intersection.

- | | |
|--------------|---------------|
| A. 965.926 m | C. 1035.276 m |
| B. 258.819 m | D. 863.103 m |

Situation:

A ship at A is to sail to C, 56 mi north and 258 mi east of A. After sailing N25°10' E for 120 mi to P, the ship is headed toward C.

18. Find the distance of P from C.

- | | |
|---------------|---------------|
| A. 156.498 mi | C. 213.551 mi |
| B. 197.446 mi | D. 253.331 mi |

19. Find the required course to reach C.

- | | |
|---------------------|---------------------|
| A. N 14°15'42.46" W | C. S 14°15'42.46" E |
| B. N 75°44'17.54" W | D. S 75°44'17.54" E |

Situation:

Three circles of radii 115, 150, and 225 m are tangent to each other externally. Angles are formed by joining the centers of the circles.

20. Find the smallest angle.

- | | |
|-----------------|-----------------|
| A. 43°09'46.43" | C. 75°28'23.61" |
| B. 61°21'41.96" | D. 59°59'57.33" |

21. Find the smaller angle.

- | | |
|-----------------|-----------------|
| A. 43°09'46.43" | C. 75°28'23.61" |
| B. 61°21'41.96" | D. 59°59'57.33" |

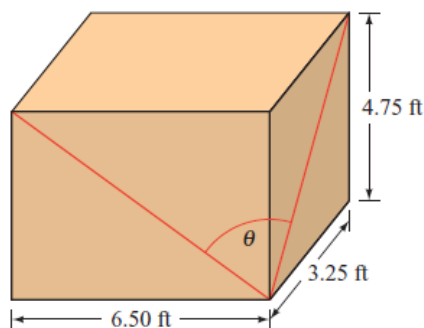
22. Find the biggest angle.

- | | |
|-----------------|-----------------|
| A. 43°09'46.43" | C. 75°28'23.61" |
| B. 61°21'41.96" | D. 59°59'57.33" |

23. A woman hikes 503 m, turns and jogs 415 m, turns again, and runs 365 m returning to her starting point. What is the area of the triangle formed by her path?

- | | |
|----------------------------|----------------------------|
| A. 74594.17 m ² | C. 79544.17 m ² |
| B. 74945.17 m ² | D. 75459.17 m ² |

24. The rectangular box in the figure measures 6.50 feet by 3.25 feet by 4.75 feet. Find the measure of the angle that is formed by the union of the diagonal shown on the front of the box and the diagonal shown on the right side of the box.



- A. 29.14° C. 75.37°
 B. 60.86° D. 14.63°

Situation:

An observer at C on a hillside measures the angles of depression of two points A and B in a horizontal plane below him. A and B are in the same direction from the observer at C, A, B are in the same vertical plane. The angle of depression of A is $36^\circ 28' 30''$, and that of B is $22^\circ 16' 0''$. If the distance from A to B is 4125.0 feet:

25. Find the distance of the observer at C directly from A.
 A. 5225.57 ft. C. 6368.08 ft.
 B. 7553.22 ft. D. 7335.22 ft.
26. Find the distance of the observer at C directly from B.
 A. 10033.89 ft. C. 8774.24 ft.
 B. 9990.68 ft. D. 8609.44 ft.
27. For three points A, B, C, in a horizontal plane, the bearing of C from A is $N 34^\circ 27' 20'' E$, the bearing of C from B is $S 72^\circ 40' 40'' E$, and the bearing of B from A is $N 15^\circ 24' 30'' E$. The distance from A to B is 2450.5 yards. Find the distance from A to C.
 A. 2265.57 yards C. 2562.87 yards
 B. 2665.44 yards D. 2225.75 yards
28. $A = 47^\circ 13' 50''$, $a = 0.20633$, $b = 0.70812$. Find B.
 A. 23.39° C. 66.61°
 B. 31.29° D. No triangle formed
29. Given $A = 47^\circ 13' 35''$, $a = 0.60631$, $b = 0.70815$. Find B.
 A. 34.996° C. 145.004°
 B. 59.019° D. No triangle formed
30. Given $A = 132^\circ 47' 20''$, $a = 0.90635$, $b = 0.70810$. Find C.
 A. 34.984° C. 12.23°
 B. 59.016° D. 77.77°
31. Three points A, B, C in a horizontal plane are so situated that the bearing of B from A is $N 17^\circ 30' 0'' E$ and the bearing of C from A is $S 24^\circ 20' 20'' E$. If the length of AB is 3210.5 yards and that of BC is 4715.0 yards, find the bearing of C from B.
 A. $N 82^\circ 25' 49'' W$ C. $N 7^\circ 34' 11'' W$
 B. $S 7^\circ 34' 11'' E$ D. $S 82^\circ 25' 49'' E$

Situation:

Two sides of a parallelogram are 11.055 feet long and 13.267 feet long, respectively; and one interior angle is $72^{\circ} 15' 30''$.

32. Find the length of the longer diagonal.
- | | |
|---------------|---------------|
| A. 19.688 ft. | C. 14.451 ft. |
| B. 20.968 ft. | D. 15.415 ft. |
33. Find the length of the shorter diagonal.
- | | |
|---------------|---------------|
| A. 19.688 ft. | C. 14.451 ft. |
| B. 20.968 ft. | D. 15.415 ft. |
34. A surveyor runs a line $N 35^{\circ} 30' 30'' E$ from A to B, the length of AB being 1246.5 feet. From B, he runs a line $S 25^{\circ} 14' 0'' E$ to C, and measures BC as 1729.6 feet long. How long is AC?
- | | |
|----------------|----------------|
| A. 1651.09 ft. | C. 1555.04 ft. |
| B. 1233.90 ft. | D. 1561.36 ft. |
35. The sides of a triangle are in the proportion 3:4:5; the area of the triangle is 108 sq. in. Find the radius of the inscribed circle.
- | | |
|-------------|-------------|
| A. 2.828 in | C. 8.485 in |
| B. 4.243 in | D. 2.121 in |

Situation:

In tunneling under a river, a tunnel AB was first made at an angle of depression of $12^{\circ}30'$, then a horizontal tunnel BC 610 ft. long, then a tunnel CD rising at an inclination of $12^{\circ}30'$, the points A and D lying in a horizontal plane. Assume that A, B, C, D lie in a vertical plane. If the maximum depth of the tunnel is 55 ft.:

36. How long is the tunnel?
- | | |
|-----------------|-----------------|
| A. 1106.178 ft. | C. 1118.225 ft. |
| B. 633.808 ft. | D. 720 ft. |
37. How far apart is A and D?
- | | |
|-----------------|-----------------|
| A. 1106.178 ft. | C. 1118.125 ft. |
| B. 633.808 ft. | D. 720 ft. |
38. What is the radius of the largest gas tank that could be placed on a triangular lot whose sides are 84.027 ft., 77.526 ft., and 102.473 ft. long respectively?
- | | |
|---------------|---------------|
| A. 24.188 ft. | C. 52.263 ft. |
| B. 33 ft. | D. 14.14 ft. |

Situation:

The angles of a triangle are $36^{\circ} 20' 20''$, $79^{\circ} 30' 40''$, and $64^{\circ} 10' 0''$; the radius of the circumscribed circle is 2.2534 in. long.

39. Find the length of the longest side.
- | | |
|-----------------|-----------------|
| A. 4.431 inches | C. 2.670 inches |
| B. 4.056 inches | D. 5.332 inches |
40. Find the area of the triangle.
- | | |
|-------------------------|-------------------------|
| A. 2.356 in^2 | C. 5.325 in^2 |
| B. 3.625 in^2 | D. 5.263 in^2 |

PLANE AND SPHERICAL TRIGONOMETRY SOLUTIONS

Situation:

A nautical mile is the length of an arc, on Earth's equator, that subtends a central angle. The equatorial radius of Earth is about 3960 statute miles.

- Convert 1 nautical mile to statute miles.

Solution:

Remember:

$$1^\circ = 60 \text{ minutes}$$

Therefore:

$$\frac{1 \text{ nautical mile}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ degree}} \cdot \frac{360^\circ}{\text{circumference}} = \frac{21600 \text{ nautical mile}}{\text{circumference}}$$

But:

$$\text{circumference} = 2\pi(3960) = 24882.414 \text{ statute miles}$$

Thus,

$$\frac{(21600 \text{ nautical mile})}{24881.414 \text{ statute mile}} = \mathbf{0.868 \text{ nautical mile}} \text{ per statute mile}$$

- Determine what percent of Earth's circumference is covered by a trip from Los Angeles, California, to Honolulu, Hawaii (2217 nautical miles).

Solution:

$$1 \text{ circumference} = 21600 \text{ nautical miles}$$

Therefore,

$$\frac{2217 \text{ nautical mile}}{21600 \text{ nautical mile}} = 0.1026 \text{ or } \mathbf{10.26\%}$$

- Which of the following is the value of x ? $\frac{1+\sin 2\theta-\cos 2\theta}{1+\sin 2\theta+\cos 2\theta} = x$

First Solution:

Using Trigonometric Identity:

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

We have:

$$\begin{aligned} x &= \frac{1 + 2 \sin \theta \cos \theta - (\cos^2 \theta - \sin^2 \theta)}{1 + 2 \sin \theta \cos \theta + (\cos^2 \theta - \sin^2 \theta)} \\ x &= \frac{1 + \sin^2 \theta + 2 \sin \theta \cos \theta - \cos^2 \theta}{1 + \cos^2 \theta + 2 \sin \theta \cos \theta - \sin^2 \theta} \\ x &= \frac{(1 - \cos^2 \theta) + \sin^2 \theta + 2 \sin \theta \cos \theta}{(1 - \sin^2 \theta) + \cos^2 \theta + 2 \sin \theta \cos \theta} \\ x &= \frac{\sin^2 \theta + \sin^2 \theta + 2 \sin \theta \cos \theta}{\cos^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta} \\ x &= \frac{2 \sin^2 \theta + 2 \sin \theta \cos \theta}{2 \cos^2 \theta + 2 \sin \theta \cos \theta} \\ x &= \frac{2 \sin \theta (\sin \theta + \cos \theta)}{2 \cos \theta (\sin \theta + \cos \theta)} \\ x &= \frac{\sin \theta}{\cos \theta} \\ x &= \mathbf{\tan \theta} \end{aligned}$$

Second solution:

Try to match the values from the choices:

- $2 \sin \theta$
- $3 \cos \theta + 2$
- $\tan \theta$
- $\sin^2 \theta$

Try $\theta = 50^\circ$:

$$x = \frac{1 + \sin 100^\circ - \cos 100^\circ}{1 + \sin 100^\circ + \cos 100^\circ}$$

$$x = 1.19175$$

Try choice a:

$$2 \sin \theta = 2 \sin 50^\circ = 1.532 \neq 1.19175$$

Try choice b:

$$3 \cos \theta + 2 = 3 \cos 50^\circ + 2 = 3.928 \neq 1.19175$$

Try choice c:

$$\tan \theta = \tan 50^\circ = 1.19175 = 1.19175$$

Try choice d:

$$\sin^2 \theta = \sin^2 50^\circ = 0.587 \neq 1.19175$$

Therefore, the answer is **c**.

4. If $\tan \theta = \frac{4}{3}$ and θ is an angle in the third quadrant, evaluate

$$\frac{\sin(180 + \theta)}{\sec(270 + \theta)} + \frac{\cos(360 - \theta)}{\csc(90 + \theta)}$$

Solution:

$$\tan \theta = \frac{4}{3}$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$

$$\theta = 53.131^\circ \rightarrow \text{Quadrant 1}$$

For θ to be in Quadrant 3:

$$\theta = 180^\circ + 53.131^\circ = 233.131^\circ$$

Therefore:

$$\begin{aligned} & \frac{\sin(180 + \theta)}{\sec(270 + \theta)} + \frac{\cos(360 - \theta)}{\csc(90 + \theta)} \\ &= \frac{\sin(180 + 233.131^\circ)}{\sec(270 + 233.131^\circ)} + \frac{\cos(360 - 233.131^\circ)}{\csc(90 + 233.131^\circ)} \\ & \frac{\sin(180 + \theta)}{\sec(270 + \theta)} + \frac{\cos(360 - \theta)}{\csc(90 + \theta)} = \frac{\sin 413.131^\circ}{\left(\frac{1}{\cos 503.131^\circ}\right)} + \frac{\cos 126.869^\circ}{\left(\frac{1}{\sin 323.131^\circ}\right)} \\ & \frac{\sin(180 + \theta)}{\sec(270 + \theta)} + \frac{\cos(360 - \theta)}{\csc(90 + \theta)} = \mathbf{-0.28} \end{aligned}$$

5. The angle of depression to one side of a lake, measured from a balloon 2500 feet above the lake as shown in the accompanying figure, is 43° . The angle of depression to the opposite side of the lake is 27° . Find the width of the lake.

Solution:

$$\tan 43^\circ = \frac{2500}{x_1}$$

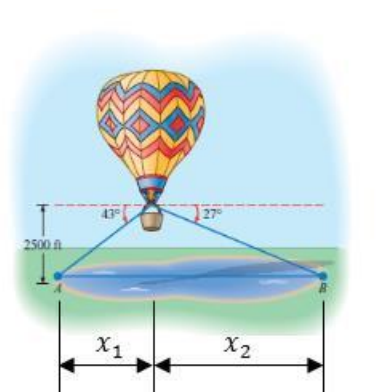
$$x_1 = \frac{2500}{\tan 43^\circ}$$

$$x_1 = 2680.921'$$

Applying the same manner:

$$x_2 = \frac{2500}{\tan 27^\circ}$$

$$x_2 = 4906.526'$$

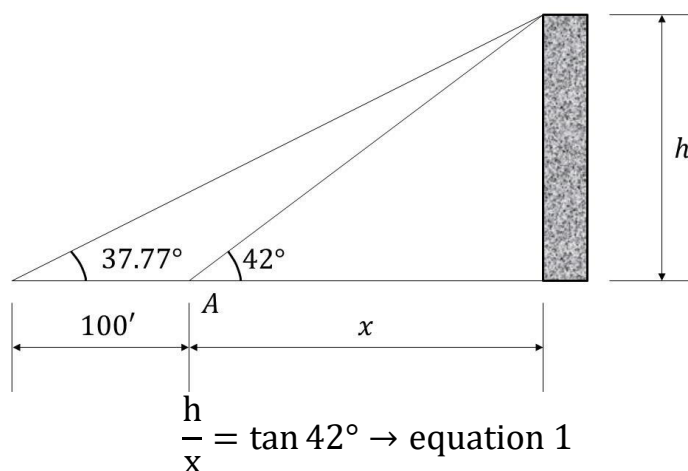


Therefore:

$$\overline{AB} = x_1 + x_2 = 2680.921 + 4906.526 = \mathbf{7587.447'}$$

6. From a point A on a line from the base of the Washington Monument, the angle of elevation to the top of the monument is 42° . From a point 100 feet away from A and on the same line, the angle to the top is 37.77° . Find the height of the Washington Monument.

Solution:



$$\frac{h}{x} = \tan 42^\circ \rightarrow \text{equation 1}$$

$$\frac{h}{x + 100} = \tan 37.77^\circ \rightarrow \text{equation 2}$$

From equation 1, we have:

$$h = x \tan 42^\circ$$

Substituting it to equation 2, we have:

$$\frac{x \tan 42^\circ}{x + 100} = \tan 37.77^\circ$$

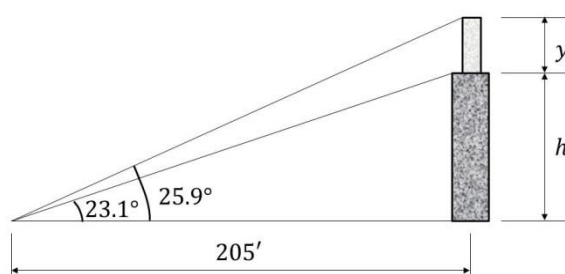
$$x = 617.094 \text{ ft}$$

$$h = x \tan 42^\circ = 617.094 \tan 42^\circ$$

$$\mathbf{h = 555.634 \text{ ft}}$$

Situation:

From point A, at street level and 205 feet from the base of a building, the angle of elevation to the top of the building is 23.1° . Also, from point A the angle of elevation to the top of a neon sign, which is at the top the building, is 25.9° .



7. Determine the height of the building.

Solution:

Referring to the figure:

$$\frac{h}{205} = \tan 23.1^\circ$$

$$\mathbf{h = 87.44 \text{ ft}}$$

8. How tall is the sign?

Solution:

Referring to the figure:

$$\frac{y + h}{205} = \tan 25.9^\circ$$

$$\frac{y + 87.44}{205} = \tan 25.9^\circ$$

$$\mathbf{y = 12.103 \text{ ft}}$$

9. A right triangle ACB with the right angle at C has legs 5 m and 12 m. Find the length of a line drawn from C perpendicular to the hypotenuse.

Solution:

Referring to the figure:

$$\text{Area}_{\Delta ACB} = \frac{1}{2}(5)(12)$$

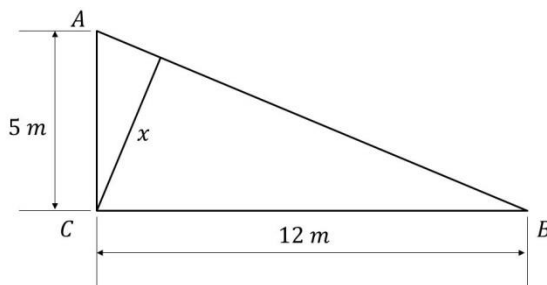
$$\text{Area}_{\Delta ACB} = 30 \text{ m}^2$$

Also:

$$\text{Area}_{\Delta ACB} = \frac{1}{2}(13)(x)$$

$$\frac{1}{2}(13)(x) = 30 \text{ m}^2$$

$$\mathbf{x = 4.615 \text{ m}}$$



10. The bases of a parcel of land in the form of a trapezoid are 92.6 m and 75.8 m, respectively. The angle at the extremities of the longer base is 72° and 43° , respectively. Find the perimeter of the parcel of land.

Solution:

Referring to the figure:

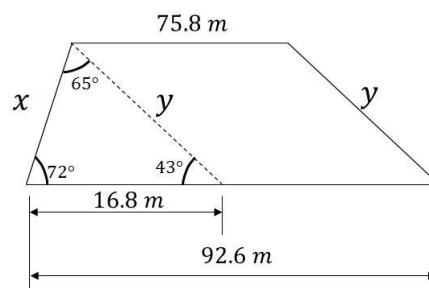
By sine law:

$$\frac{x}{\sin 43^\circ} = \frac{16.8}{\sin 65^\circ}$$

$$x = 12.642 \text{ m}$$

$$\frac{y}{\sin 72^\circ} = \frac{16.8}{\sin 65^\circ}$$

$$y = 17.629 \text{ m}$$



$$P = x + 75.8 + y + 92.6$$

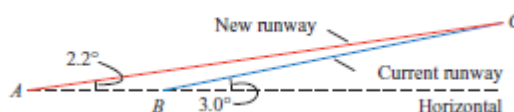
$$P = 12.642 + 75.8 + 17.629 + 92.6$$

$$\mathbf{P = 198.672 \text{ m}}$$

11. An airport runway is 3550 feet long and has an incline of 3.0° . The airport planning committee plans to replace this runway with a new runway, as shown in the following figure. The new runway will be inclined at an angle of 2.2° . What will be the length of the new runway?

Solution:

Referring to the figure:



By sine law:

$$\frac{L}{\sin 177^\circ} = \frac{3550}{\sin 2.2^\circ}$$

$$\mathbf{L = 4839.887 \text{ ft}}$$

12. Two observers, in the same vertical plane and 30 feet apart, observe the kite at angles of 72° and 78° , respectively. Find the height of the kite.

Solution:

Referring to the figure:

$$\frac{h}{x} = \tan 72^\circ \rightarrow \text{equation 1}$$

$$\frac{h}{30 - x} = \tan 78^\circ \rightarrow \text{equation 2}$$

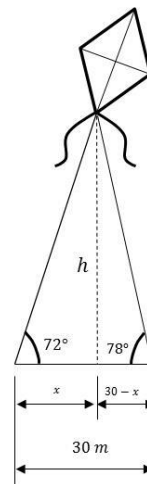
From equation 1, we have:

$$h = x \tan 72^\circ$$

Substituting to equation 2:

$$\frac{x \tan 72^\circ}{30 - x} = \tan 78^\circ$$

$$x = 18.136 \text{ ft}$$



Therefore:

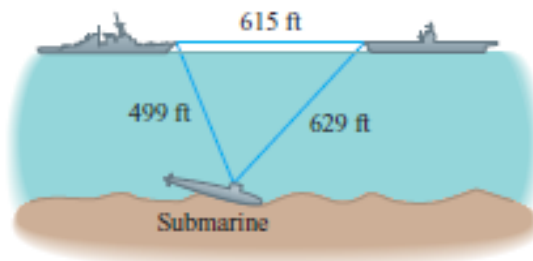
$$h = x \tan 72^\circ$$

$$h = 18.136 \tan 72^\circ$$

$$\mathbf{h = 55.816 \text{ ft}}$$

13. Use the distances shown in the following figure to determine the depth of the submarine below the surface of the water. Assume that the line segment between the surface ships is directly above the submarine.

Solution:



Using cosine law:

$$499^2 + 615^2 - 2(499)(615) \cos \theta = 629^2$$

Solving for θ :

$$\cos \theta = \frac{499^2 + 615^2 - 629^2}{2(499)(615)}$$

$$\cos \theta = 0.3773$$

$$\theta = 67.83^\circ$$

$$h = 499 \sin \theta$$

$$h = 499 \sin 67.83^\circ$$

$$\mathbf{h = 462.116 \text{ ft}}$$

14. A right triangle with sides 4.32 and 2.41 inches long respectively is inscribed in a circle. What is the diameter of the circle?

Solution:

If a right triangle is inscribed in a circle, the hypotenuse of the triangle is the diameter of the circle.

By Pythagorean Theorem:

$$\overline{AC}^2 = \sqrt{4.32^2 + 2.41^2}$$

$$\overline{AC} = \mathbf{4.947 \text{ in}}$$

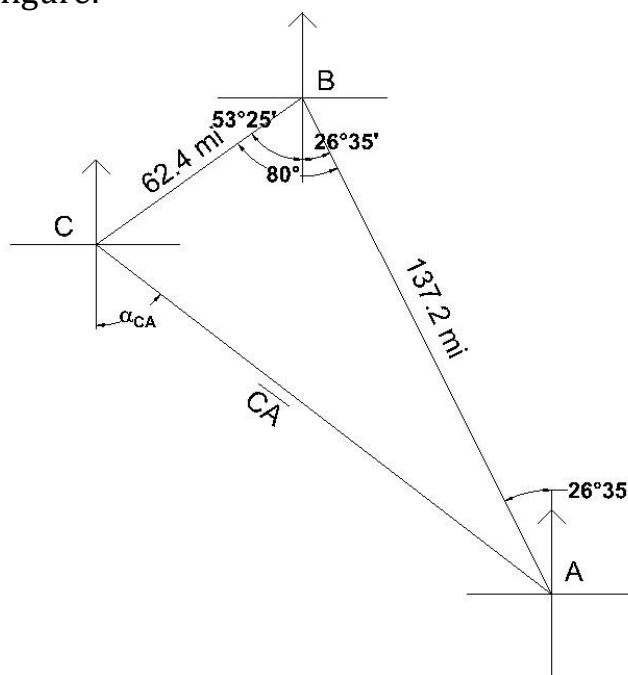
Situation:

An aero plane flies at N 26°35' W for 137.2 miles, then S 53°25' W for 62.4 miles.

15. How far must it go?

Solution:

Referring to the figure:



By cosine law:

$$\overline{CA}^2 = \overline{BC}^2 + \overline{BA}^2 - 2(\overline{BC})(\overline{BA}) \cos \theta$$

Where:

$$\overline{BC} = 62.4 \text{ miles}$$

$$\overline{BA} = 137.2 \text{ miles}$$

$$\theta = 26^\circ 35' + 53^\circ 25' = 80^\circ$$

Therefore:

$$\overline{CA}^2 = 62.4^2 + 137.2^2 - 2(62.4)(137.2) \cos 80^\circ$$

$$\overline{CA} = \mathbf{140.514 \text{ miles}}$$

16. In what direction should it then fly to return to the starting point in a straight line?

Solution:

Referring to the figure:

By sine law:

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\frac{\sin C}{137.2} = \frac{\sin 80^\circ}{140.514}$$

$$C = 74^\circ 03' 57.5''$$

Therefore, the bearing is:

$$\alpha_{CA} = 180^\circ - 74^\circ 03' 57.5'' - 53^\circ 25'$$

$$\alpha_{CA} = \mathbf{S 52^\circ 31' 02.5'' E}$$

17. Two straight roads intersect to form an angle of 75°. Find the shortest distance from one road to a gas station on the other road that is 1000 m from the intersection.

Solution:

Referring to the figure:

Since the shortest distance is the perpendicular distance, we have:

$$\sin 75^\circ = \frac{d}{1000 \text{ m}}$$

$$d = 1000 \sin 75^\circ$$

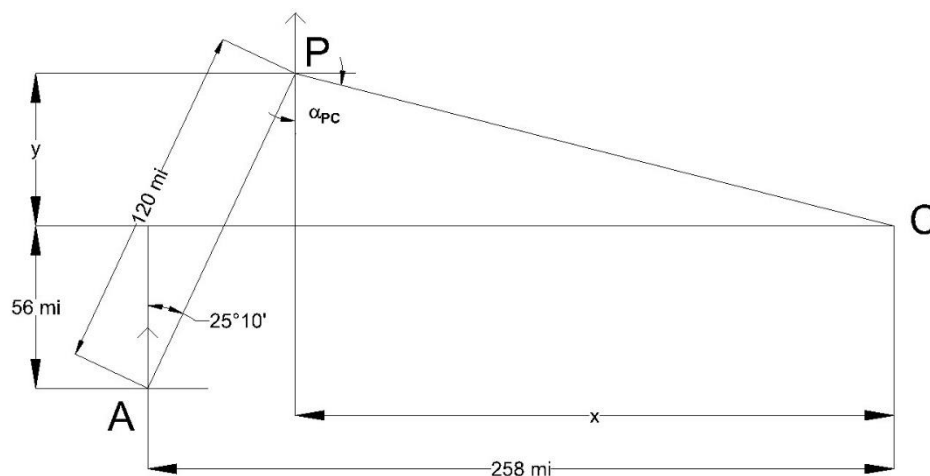
$$d = \mathbf{965.926 \text{ m}}$$

Situation:

A ship at A is to sail to C, 56 km north and 258 km east of A. After sailing N25°10' E for 120 mi to P, the ship is headed toward C.

18. Find the distance of P from C.

Solution:



Referring to the figure:

$$120 \sin(25^\circ 10') + x = 258$$

$$x = 206.970 \text{ km} \rightarrow \text{STO X}$$

$$120 \cos(25^\circ 10') = 56 + y$$

$$y = 52.609 \text{ km} \rightarrow \text{STO Y}$$

By Pythagorean Theorem:

$$L^2 = \sqrt{X^2 + Y^2}$$

$$L^2 = 45604.143$$

$$L = \mathbf{213.551 \text{ mi}}$$

19. Find the required course to reach C.

Solution:

$$\alpha_{PC} = \tan^{-1} \left(\frac{X}{Y} \right)$$

$$\alpha_{PC} = \mathbf{S 75^\circ 44' 17.54'' E}$$

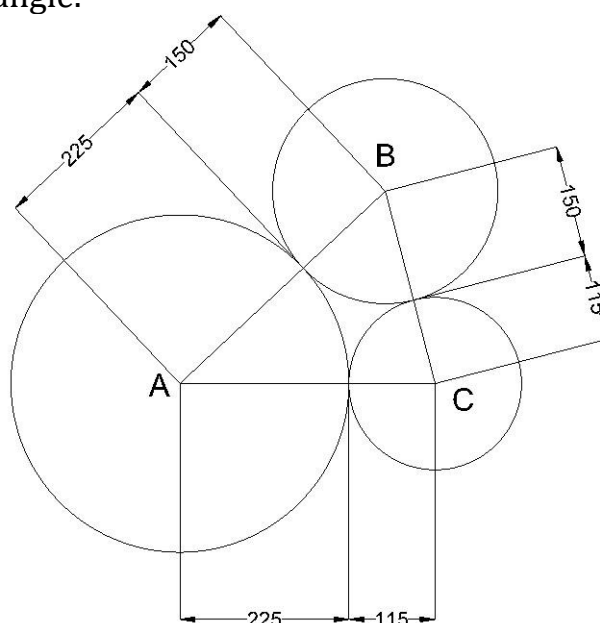
Situation:

Three circles of radii 115, 150, and 225 m are tangent to each other externally. Angles are formed by joining the centers of the circles.

20. Find the smallest angle.

21. Find the smaller angle.

22. Find the biggest angle.



Solution:

Referring to the figure:

By cosine law:

$$265^2 + 375^2 - 2(265)(375) \cos B = 340^2$$

Solving for B:

$$\cos B = \frac{265^2 + 375^2 - 340^2}{2(265)(375)}$$

$$\cos B = 0.479$$

$$\mathbf{B = 61^{\circ}21'41.96''}$$

By sine law:

$$\frac{\sin A}{265} = \frac{\sin 61^{\circ}21'41.96''}{340}$$

$$\mathbf{A = 43^{\circ}09'46.43''}$$

Since the sum of the angles in a triangle is equal to 180°:

$$A + B + C = 180^{\circ}$$

$$43^{\circ}09'46.43'' + 61^{\circ}21'41.96'' + C = 180^{\circ}$$

$$\mathbf{C = 75^{\circ}28'23.61''}$$

23. A woman hikes 503 m, turns and jogs 415 m, turns again, and runs 365 m returning to her starting point. What is the area of the triangle formed by her path?

Solution:

Referring to the figure:

Using Heron's formula:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

Where:

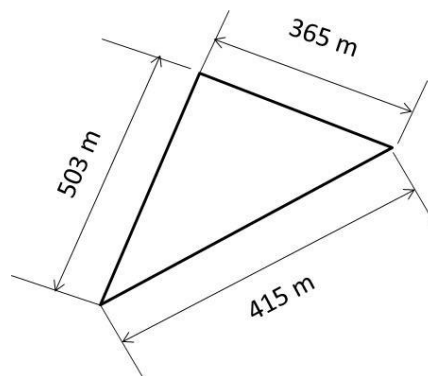
$$s = \frac{a + b + c}{2}$$

We have:

$$s = \frac{503 + 415 + 365}{2} = 641.5$$

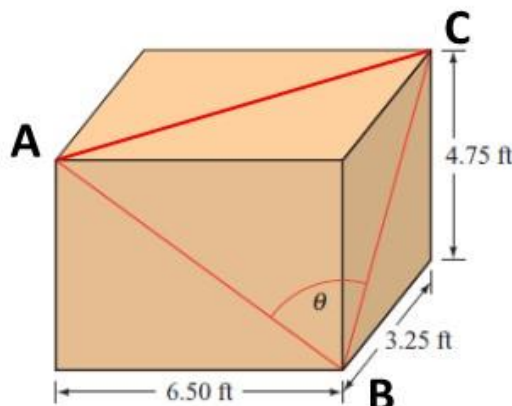
$$\text{Area} = \sqrt{641.5(641.5 - 503)(641.5 - 415)(641.5 - 365)}$$

$$\mathbf{\text{Area} = 74594.17 \text{ m}^2}$$



24. The rectangular box in the figure measures 6.50 feet by 3.25 feet by 4.75 feet. Find the measure of the angle that is formed by the union of the diagonal shown on the front of the box and the diagonal shown on the right side of the box.

Solution:



Referring to the figure:

$$\overline{AB} = \sqrt{4.75^2 + 6.5^2} = 8.05 \text{ ft}$$

$$\overline{BC} = \sqrt{3.25^2 + 4.75^2} = 5.755 \text{ ft}$$

$$\overline{AC} = \sqrt{6.5^2 + 3.25^2} = 7.267 \text{ ft}$$

By cosine law:

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2(\overline{AB})(\overline{BC}) \cos \theta$$

$$7.267^2 = 8.05^2 + 5.755^2 - 2(8.05)(5.755) \cos \theta$$

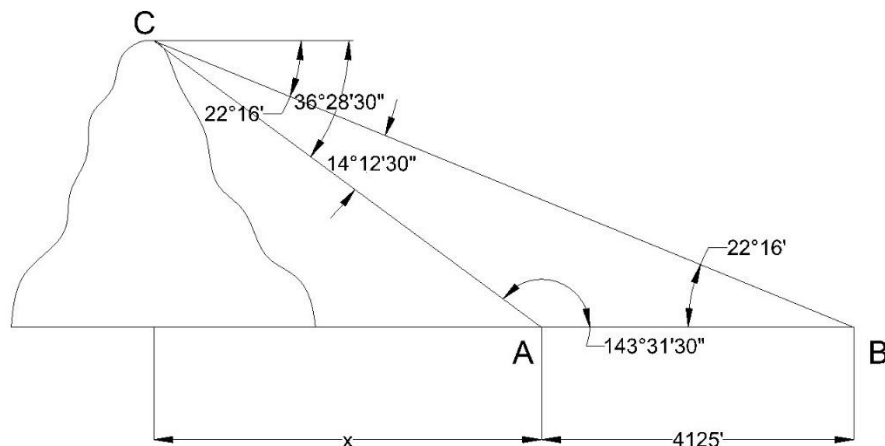
Solving for θ :

$$\cos \theta = 0.48694$$

$$\theta = 60.86^\circ$$

Situation:

An observer at C on a hillside measures the angles of depression of two points A and B in a horizontal plane below him. A and B are in the same direction from the observer at C, A, B are in the same vertical plane. The angle of depression of A is $36^\circ 28' 30''$, and that of B is $22^\circ 16' 0''$. If the distance from A to B is 4125.0 feet:



25. Find the distances of C from A.

Solution:

By sine law:

$$\frac{AC}{\sin(22^\circ 16')} = \frac{4125}{\sin(14^\circ 12' 30'')}$$

$$AC = 6368.985 \text{ ft}$$

26. Find the distances of C from B.

Solution:

By sine law:

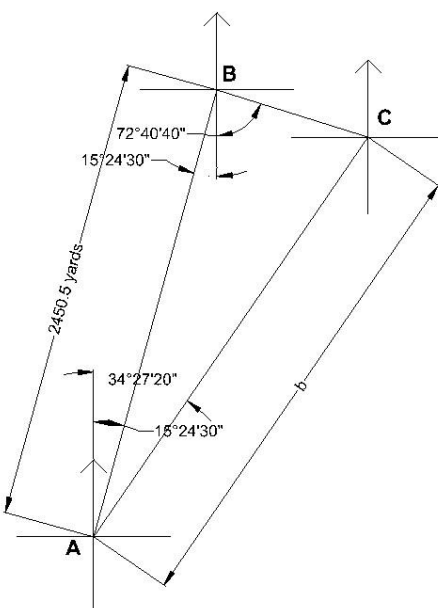
$$\frac{BC}{\sin(143^\circ 31' 30'')} = \frac{4125}{\sin(14^\circ 12' 30'')}$$

$$BC = 9990.683 \text{ ft}$$

27. For three points A, B, C, in a horizontal plane, the bearing of C from A is $N 34^\circ 27' 20'' E$, the bearing of C from B is $S 72^\circ 40' 40'' E$, and the bearing of B from A is $N 15^\circ 24' 30'' E$. The distance from A to B is 2450.5 yards. Find the distance from A to C.

Solution:

Referring to the figure:



Since the sum of the angles in a triangle is equal to 180° :

$$\begin{aligned} A + B + C &= 180^\circ \\ 19^\circ 02' 50'' + 88^\circ 05' 10'' + C &= 180^\circ \\ C &= 75^\circ 52' \end{aligned}$$

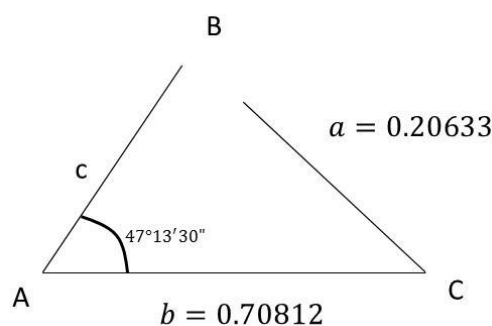
From sine law:

$$\begin{aligned} \frac{b}{\sin B} &= \frac{c}{\sin C} \\ \frac{b}{\sin 88^\circ 05' 10''} &= \frac{2450.5}{\sin 75^\circ 52'} \\ \mathbf{b} &= \mathbf{2562.87 \text{ yards}} \end{aligned}$$

28. $A = 47^\circ 13' 50''$, $a = 0.20633$, $b = 0.70812$. Find B.

Solution:

Referring to the figure:



From sine law:

$$\begin{aligned} \frac{\sin B}{b} &= \frac{\sin A}{a} \\ \frac{\sin B}{0.70812} &= \frac{\sin 47^\circ 13' 50''}{0.20633} \\ \sin B &= 2.519 \end{aligned}$$

Since $\sin B > 1$:

Therefore, there is **no triangle formed**.

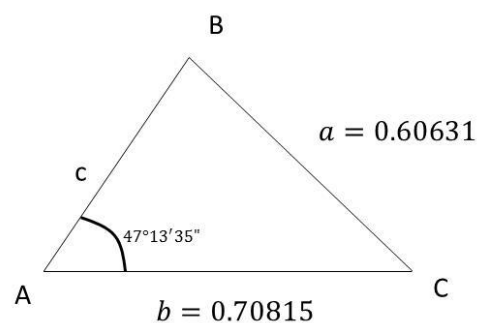
29. Given $A = 47^\circ 13' 35''$, $a = 0.60631$, $b = 0.70815$. Find B.

Solution:

Referring to the figure:

From sine law:

$$\begin{aligned} \frac{\sin B}{b} &= \frac{\sin A}{a} \\ \frac{\sin B}{0.70815} &= \frac{\sin 47^\circ 13' 35''}{0.60631} \\ \sin B &= 0.8573 \\ B &= 59.019^\circ \text{ or} \\ B &= 180^\circ - 59.019 = 120.981^\circ \end{aligned}$$



Since,

$$120.981^\circ + 47^\circ 13' 35'' = 168.27^\circ < 180^\circ$$

Hence, there are two triangles formed. Based on the choices, the answer is **59.019°** .

30. Given $A = 132^\circ 47' 20''$, $a = 0.90635$, $b = 0.70810$. Find C .

Solution:

Referring to the figure:

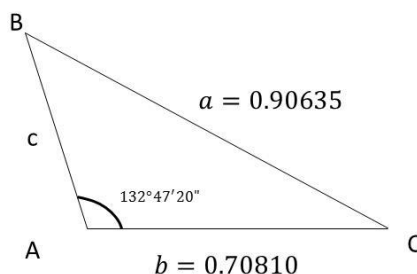
By sine law:

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\frac{\sin B}{0.70810} = \frac{\sin 132^\circ 47' 20''}{0.90635}$$

$$B = 34.984^\circ \text{ or}$$

$$B = 180^\circ - 34.984^\circ = 145.016^\circ$$



Since,

$$145.016^\circ + 132^\circ 47' 20'' = 277^\circ 48' 19.34'' > 180^\circ$$

Therefore, the only value is:

$$B = 34.984^\circ$$

Since the sum of the angles in a triangle is equal to 180° :

$$A + B + C = 180^\circ$$

$$132^\circ 47' 20'' + 34.984^\circ + C = 180^\circ$$

$$C = 12.23^\circ$$

31. Three points A, B, C in a horizontal plane are so situated that the bearing of B from A is $N 17^\circ 30' 0'' E$ and the bearing of C from A is $S 24^\circ 20' 20'' E$. If the length of AB is 3210.5 yards and that of BC is 4715.0 yards, find the bearing of C from B.

Solution:

Referring to the figure:

Using cosine law:

$$a^2 = b^2 + c^2 - 2bc \cos \theta$$

$$a^2 = 4715^2 + 3210.5^2 - 2(4715)(3210.5) \cos 138^\circ 09' 40''$$

$$a = 7422.542 \text{ yards}$$

Cosine law can be written in the form:

Go to Complex MODE:

Then type:

$$|4715 - 3210.5 \angle 138^\circ 09' 40''| = 7422.542 \text{ yards or}$$

$$|3210.5 - 4715 \angle 138^\circ 09' 40''| = 7422.542 \text{ yards}$$

By sine law:

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

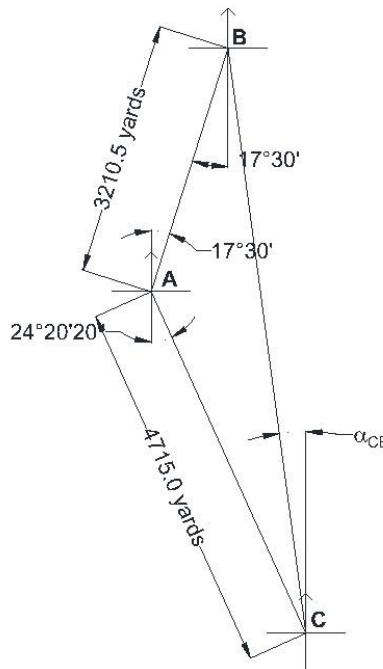
$$\frac{\sin C}{3210.5} = \frac{\sin 138^\circ 09' 40''}{7422.542}$$

$$C = 16^\circ 46' 09''$$

Therefore,

$$\alpha_{CB} = 90^\circ - (16^\circ 46' 09'' + 65^\circ 39' 40'')$$

$$\alpha_{CB} = N 7^\circ 34' 11'' W$$



Situation:

Two sides of a parallelogram are 11.055 feet long and 13.267 feet long, respectively; and one interior angle is $72^\circ 15' 30''$.

32. Find the length of the longer diagonal.

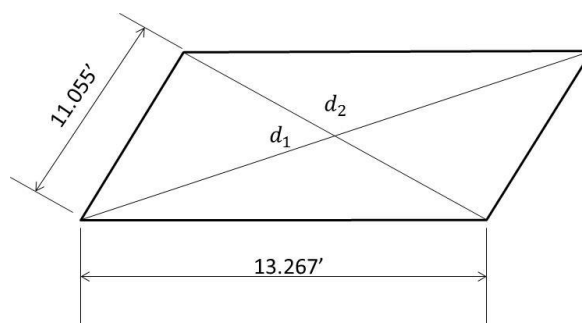
Solution:

Referring to the figure:

The adjacent angles of a parallelogram are supplements.

$$\alpha = 180^\circ - 72^\circ 15' 30''$$

$$\alpha = 107^\circ 44' 30''$$



By cosine law:

$$d_1 = \sqrt{13.267^2 + 11.055^2 - 2(13.267)(11.055) \cos 107^\circ 44' 30''}$$

$$d_1 = \mathbf{19.688 \text{ ft}}$$

Or by using Complex MODE:

$$d_1 = |13.267 - 11.055 \angle 107^\circ 44' 30''| = 19.688 \text{ ft or}$$

$$d_1 = |11.055 - 13.267 \angle 107^\circ 44' 30''| = \mathbf{19.688 \text{ ft}}$$

33. Find the length of the shorter diagonal.

Solution:

$$d_2 = \sqrt{11.055^2 + 13.267^2 - 2(11.055)(13.267) \cos 72^\circ 15' 30''}$$

$$d_2 = \mathbf{14.451 \text{ ft}}$$

Or using Complex MODE:

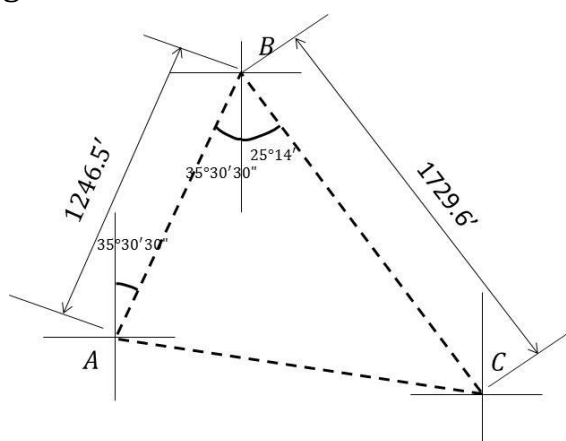
$$d_2 = |11.055 - 13.267 \angle 72^\circ 15' 30''| = 14.451 \text{ ft or}$$

$$d_2 = |13.267 - 11.055 \angle 72^\circ 15' 30''| = \mathbf{14.451 \text{ ft}}$$

34. A surveyor runs a line $N 35^\circ 30' 30'' E$ from A to B, the length of AB being 1246.5 feet. From B, he runs a line $S 25^\circ 14' 0'' E$ to C, and measures BC as 1729.6 feet long. How long is AC?

Solution:

Referring to the figure:



Using cosine law:

$$\overline{AC} = \sqrt{1246.5^2 + 1729.6^2 - 2(1246.5)(1729.6) \cos 60^\circ 44' 30''}$$

$$\overline{AC} = \mathbf{1561.361 \text{ ft}}$$

Or using Complex MODE:

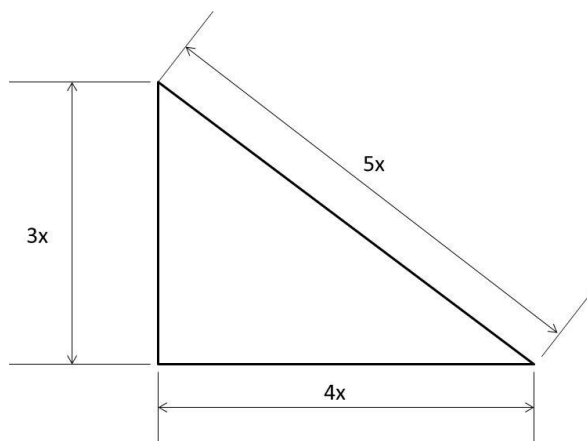
$$\overline{AC} = |1246.5 - 1729.6 \angle 60^\circ 44' 30''| = 1561.361 \text{ ft or}$$

$$\overline{AC} = |1729.6 - 1246.5 \angle 60^\circ 44' 30''| = \mathbf{1561.361 \text{ ft}}$$

35. The sides of a triangle are in the proportion 3:4:5; the area of the triangle is 108 sq. in. Find the radius of the inscribed circle.

Solution:

Referring to the figure:



$$A = 108 \text{ in}^2$$

By Heron's Formula:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

$$s = \frac{a+b+c}{2} = \frac{3x+4x+5x}{2} = 6x$$

We have,

$$A = \sqrt{6x(6x-3x)(6x-4x)(6x-5x)}$$

$$A = \sqrt{6x(3x)(2x)(x)}$$

$$A = \sqrt{36x^4}$$

$$A = 6x^2$$

Substituting:

$$6x^2 = 108$$

$$x = 3\sqrt{2} \text{ in}$$

Therefore:

$$s = 6x = 6(3\sqrt{2}) = 18\sqrt{2} \text{ in}$$

We have:

$$r = \frac{A}{s}$$

$$r = \frac{108}{18\sqrt{2}}$$

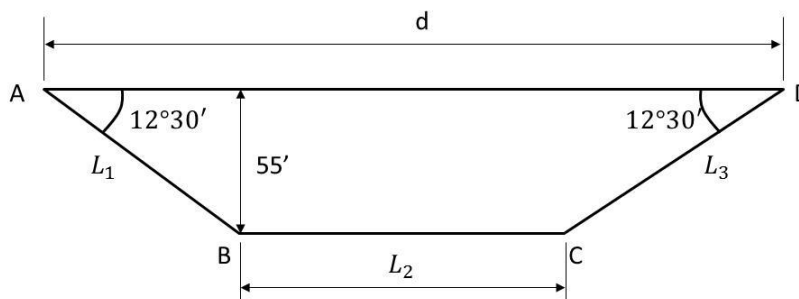
$$r = 3\sqrt{2} = 4.243 \text{ in}$$

Situation:

In tunneling under a river, a tunnel AB was first made at an angle of depression of $12^\circ 30'$, then a horizontal tunnel BC 610 ft. long, then a tunnel CD rising at an inclination of $12^\circ 30'$, the points A and D lying in a horizontal plane. Assume that A, B, C, D lie in a vertical plane. If the maximum depth of the tunnel is 55 ft.:

36. How long is the tunnel?

Solution:



Referring to the figure:

$$\overline{AB} \sin 12^\circ 30' = 55$$

$$\overline{AB} = 254.112 \text{ ft}$$

$$\overline{CD} \sin 12^\circ 30' = 55$$

$$\overline{CD} = 254.112 \text{ ft}$$

Therefore,

$$L = \overline{AB} + \overline{BC} + \overline{CD}$$

$$L = 254.112 + 610 + 254.112$$

$$L = \mathbf{1118.225 \text{ ft}}$$

37. How far apart is A and D?

Solution:

$$d = \overline{AB} \cos 12^\circ 30' + 610 + \overline{CD} \cos 12^\circ 30'$$

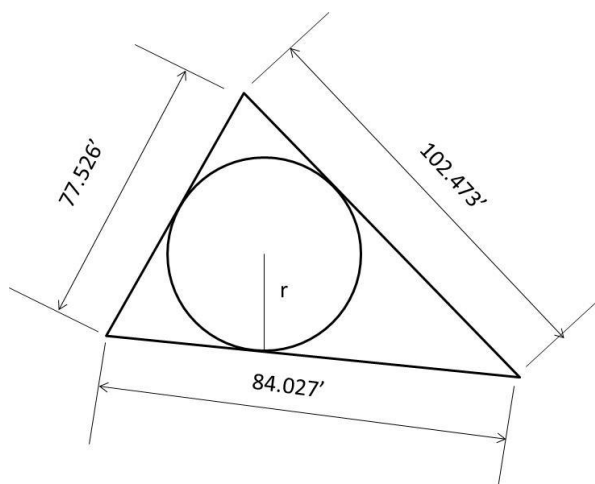
$$d = 254.112 \cos 12^\circ 30' + 610 + 254.112 \cos 12^\circ 30'$$

$$d = \mathbf{1106.178 \text{ ft}}$$

38. What is the radius of the largest gas tank that could be placed on a triangular lot whose sides are 84.027 ft, 77.526 ft, and 102.473 ft. long respectively?

Solution:

Referring to the figure:



$$r = \frac{A_T}{s}$$

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

$$s = \frac{a+b+c}{2}$$

$$s = \frac{84.027 + 77.526 + 102.473}{2}$$

$$s = 132.01 \text{ ft}$$

$$A = \sqrt{132.01(132.01 - 84.027)(132.01 - 77.526)(132.01 - 102.473)}$$

$$A = 3193.135 \text{ ft}^2$$

Therefore:

$$r = \frac{3193.135}{132.013}$$

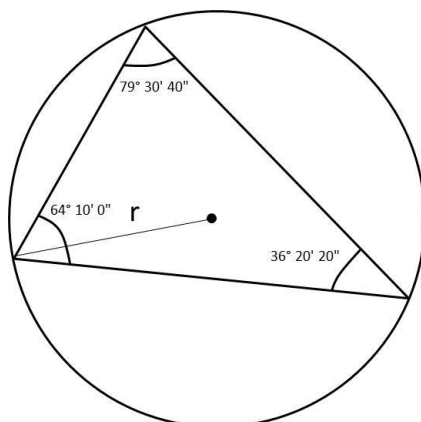
$$r = \mathbf{24.188 \text{ ft}}$$

Situation:

The angles of a triangle are $36^\circ 20' 20''$, $79^\circ 30' 40''$, and $64^\circ 10' 0''$; the radius of the circumscribed circle is 2.2534 in. long.

39. Find the length of the longest side.

Solution:



Referring to the figure:

Given that:

$$r_c = 2.2534 \text{ in}$$

From the formula:

$$A = \frac{abc}{4r_c}$$

Rearranging,

$$r_c = \frac{abc}{4A}$$

$$A = \frac{1}{2} ab \sin 64^\circ 10'$$

Therefore,

$$2.2534 = \frac{abc}{4 \left(\frac{1}{2} ab \sin 64^\circ 10' \right)}$$

$$2.2534 = \frac{c}{2 \sin 64^\circ 10'}$$

$$c = 4.056 \text{ in}$$

From sine law:

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 36^\circ 20' 20''} = \frac{4.056}{\sin 64^\circ 10'}$$

$$a = 2.671 \text{ in}$$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\frac{b}{\sin 79^\circ 30' 40''} = \frac{2.671}{\sin 36^\circ 20' 20''}$$

$$b = 4.431 \text{ in}$$

Therefore, the answer is **4.431 inches**.

40. Find the area of the triangle.

Solution:

$$A = \frac{1}{2} ab \sin C$$

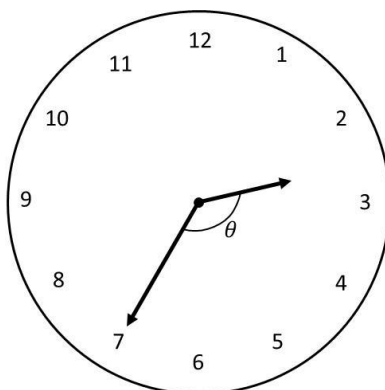
$$A = \frac{1}{2} (2.671)(4.431) \sin 64^\circ 10'$$

$$A = 5.326 \text{ in}^2$$

41. The hands of a clock are 2.250 ft. and 1.725 ft. long respectively. How far apart are their tips when the time is 2:35?

Solution:

Referring to the figure:



The angle between the minute hand and hour hand is:

$$\theta = 35(6^\circ) - 60^\circ - 35(0.5^\circ) = 132.5^\circ$$

By cosine law:

$$d = \sqrt{2.25^2 + 1.725^2 - 2(2.25)(1.725) \cos 132.5^\circ}$$

$$d = \mathbf{3.645 \text{ ft}}$$

Or using Complex MODE:

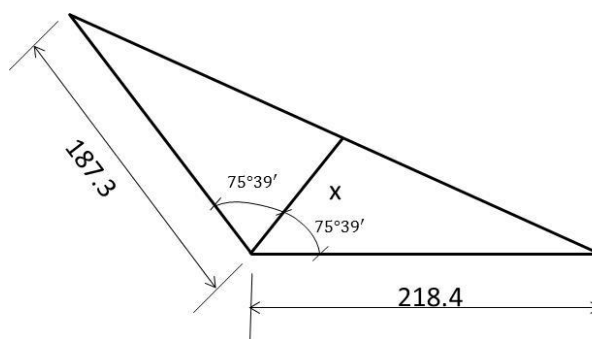
$$d = |2.25 - 1.725 \angle 132.5^\circ| = 3.645 \text{ ft or}$$

$$d = |1.725 - 2.25 \angle 132.5^\circ| = \mathbf{3.645 \text{ ft}}$$

42. Two sides of a triangle are 187.3 and 218.4, and the angle between them is $151^\circ 18'$. Find the lengths of the segments into which the opposite side is divided by the bisector of this angle.

Solution:

Referring to the figure:



$$A_T = \frac{1}{2} (187.3)(218.4) \sin 151^\circ 18'$$

$$A_T = 9822.088 \text{ square units}$$

$$A_1 = \frac{1}{2} (187.3)(x) \sin 75^\circ 39' = 90.728x$$

$$A_2 = \frac{1}{2} (218.4)(x) \sin 75^\circ 39' = 105.793x$$

Therefore,

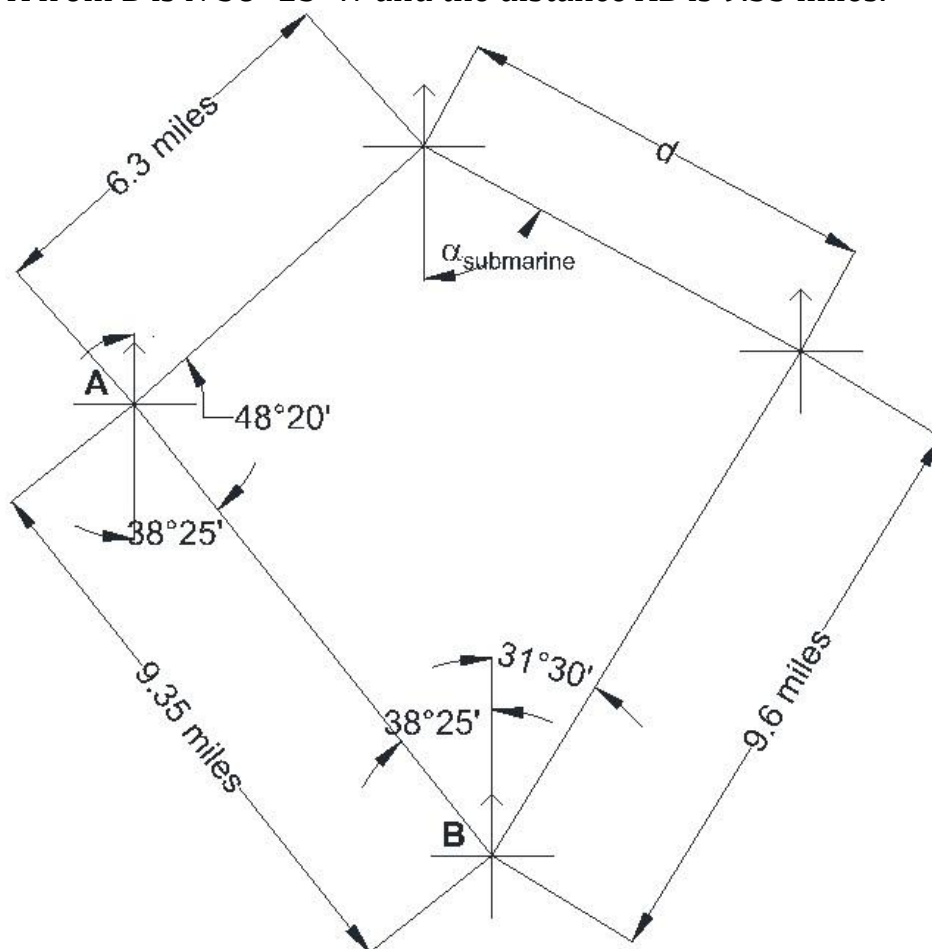
$$A_T = A_1 + A_2$$

$$9822.088 = 90.728x + 105.793x$$

$$x = \mathbf{49.980 \text{ units}}$$

Situation:

Submarine is sailing N 48° 20' E at the rate of 21 miles per hour from a point A. A chaser is sailing N 31° 30' E at the rate of 32 miles per hour from a point B. The bearing of A from B is N 38° 25' W and the distance AB is 9.35 miles.



43. How far apart will they be after 18 minutes?

Solution:

Distance traveled by the submarine in 18 minutes:

$$21\text{mph} \left(\frac{1\text{ hr}}{60\text{ min}} \right) (18\text{ min}) = 6.3\text{ miles}$$

Distance traveled by the chaser in 18 minutes:

$$32\text{mph} \left(\frac{1\text{ hr}}{60\text{ min}} \right) (18\text{ min}) = 9.6\text{ miles}$$

Referring to the figure:

Using Complex MODE:

Then type:

$$6.3\angle(270^\circ - 48^\circ 20') + 9.35\angle(270^\circ + 38^\circ 25') + 9.6\angle(90^\circ - 31^\circ 30')$$

Then press =

The answer will be:

$$6.966\angle - 28.54^\circ$$

Therefore, the answer is **6.966 miles**.

44. What will then be the bearing of the submarine from the chaser?

Solution:

Since we are looking for the bearing of the submarine, we must negate the result we get from the previous problem.

Therefore,

$$-\text{Ans} \rightarrow r\angle\theta = 6.966\angle 151.46^\circ$$

To get the angle, get the argument:

Press SHIFT-2-1:

$$\arg(\text{Ans}) = 151.46^\circ$$

This angle is measured from the positive x-axis.

Therefore,

$$\alpha_{\text{submarine}} = 151.46^\circ - 90^\circ$$

$$\alpha_{\text{submarine}} = \mathbf{N61^\circ 27' 22.69'' W}$$

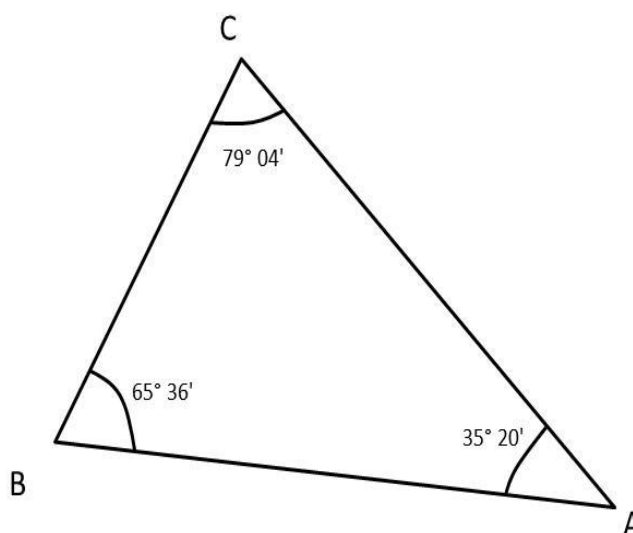
Situation:

The angles of a triangle are $A = 35^\circ 20'$, $B = 65^\circ 36'$, and $C = 79^\circ 04'$. If its area is 1200 m^2 :

45. What is the length of the side opposing A?
46. What is the length of the side opposing B?
47. What is the length of the side opposing C?

Solution:

Referring to the figure:



$$A = \frac{1}{2} bc \sin A$$

$$1200 = \frac{1}{2} bc \sin 35^\circ 20' \rightarrow \text{equation 1}$$

By sine law:

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin 79^\circ 04'}{b} = \frac{\sin 65^\circ 36'}{c}$$

$$b = c \cdot \frac{\sin 79^\circ 04'}{\sin 65^\circ 36'} \rightarrow \text{equation 2}$$

Substituting equation 2 to equation 1:

$$1200 = \frac{1}{2} \left(c \cdot \frac{\sin 79^\circ 04'}{\sin 65^\circ 36'} \right) c \sin 35^\circ 20'$$

Solving for c:

$$c = 62.04 \text{ m}$$

Substituting the value of c to equation 2:

$$b = 62.04 \cdot \frac{\sin 79^\circ 04'}{\sin 65^\circ 36'}$$

$$b = 66.89 \text{ m}$$

To solve for a:

By sine law:

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

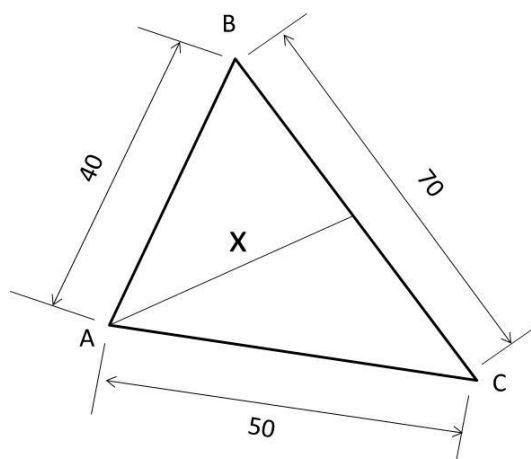
$$\frac{a}{\sin 35^\circ 20'} = \frac{62.04}{\sin 65^\circ 36'}$$

$$a = 39.40 \text{ m}$$

48. Find the length of the median to the longest side of the triangle whose sides are 40, 50, and 70, respectively.

Solution:

Referring to the figure:



By cosine law:

$$50^2 = 40^2 + 70^2 - 2(40)(70) \cos B$$

Solving for B:

$$B = 44.415^\circ$$

Again, by cosine law:

$$x^2 = 40^2 + 35^2 - 2(40)(35) \cos B$$

$$x^2 = 40^2 + 35^2 - 2(40)(35) \cos 44.415^\circ$$

$$x^2 = 825$$

$$x = 5\sqrt{33} \text{ units}$$

$$x \approx \mathbf{28.723 \text{ units}}$$

49. Given a spherical triangle: $A = 62^\circ$, $B = 49^\circ$, $a = 44^\circ$, $b = ?$

Solution:

Using sine law for spherical trigonometry:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

We have:

$$\frac{\sin 44^\circ}{\sin 62^\circ} = \frac{\sin b}{\sin 49^\circ}$$

$$\sin b = 0.593767$$

$$\mathbf{b = 36.42^\circ}$$

50. The sides a, b, and c of a spherical triangle are 80° , 140° , and 120° . Find angle A.

Solution:

Using cosine law of sides for spherical trigonometry:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

We have:

$$\cos 80^\circ = \cos 140^\circ \cos 120^\circ + \sin 140^\circ \sin 120^\circ \cos a$$

$$\mathbf{a = 112.09^\circ}$$

51. Given: $A = 112.09^\circ$, $C = 125.43^\circ$, $b = 140^\circ$, find side c .

Solution:

We cannot get directly the value of side c ; however, we must solve first for the angle B .

Using cosine law of angles for spherical trigonometry:

$$\cos B = -\cos A \cos C + \sin A \sin C \cos b$$

$$\cos B = -\cos 112.09^\circ \cos 125.43^\circ + \sin 112.09^\circ \sin 125.43^\circ \cos 140^\circ$$

$$B = 142.79^\circ$$

Next, we can use sine law to solve for the unknown, c .

$$\frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

$$\frac{\sin 140^\circ}{\sin 142.79^\circ} = \frac{\sin c}{\sin 125.43^\circ}$$

$$c = 60.00^\circ, 120.00^\circ$$

Check if these two possible solutions is really a solution.

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$

$$\cos C = -\cos 112.09^\circ \cos 142.79^\circ + \sin 112.09^\circ \sin 142.79^\circ \cos 60.00^\circ$$

$$C = 91.11^\circ$$

$$\cos C = -\cos 112.09^\circ \cos 142.79^\circ + \sin 112.09^\circ \sin 142.79^\circ \cos 120.00^\circ$$

$$C = 125.43^\circ$$

Therefore, there is only one solution to this problem.

$$\mathbf{C = 120^\circ}$$

52. The coordinates of the Dominion Astrophysical Observatory, near Victoria, British Columbia, are Latitude $48^\circ 31.3' N$ Longitude $123^\circ 25.0' W$, and the coordinates of the David Dunlap Observatory, near Toronto, Ontario, are Latitude $43^\circ 51.8' N$ Longitude $79^\circ 25.3' W$. How far is Toronto from Victoria? Express your answer in nautical miles.

Solution:

Take note:

- Latitudes are measured with the equator as reference.
- Longitudes are measured with the Prime Meridian as reference.

Draw first the polar triangle (spherical triangle) containing the North Pole or the South Pole.

Refer to the figure below:

$$b = 90^\circ - 43^\circ 51.8' = 46^\circ 8.2'$$

$$c = 90^\circ - 48^\circ 31.3' = 41^\circ 28.7'$$

$$A = 123^\circ 25.0' - 79^\circ 25.3' = 43^\circ 59.7'$$

Using cosine law of sides for spherical trigonometry:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

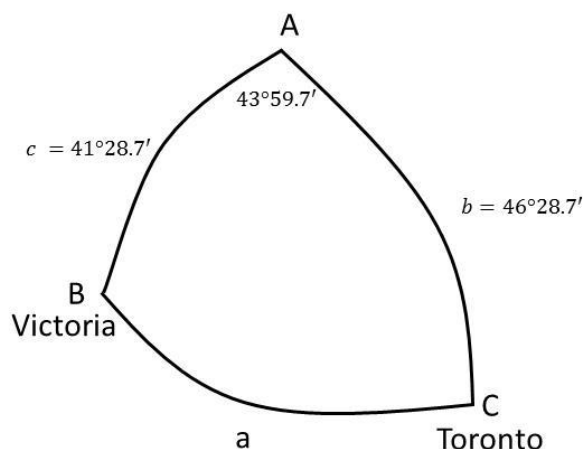
$$\cos a = \cos 46^\circ 8.2' \cos 41^\circ 28.7' + \sin 46^\circ 8.2' \sin 41^\circ 28.7' \cos 43^\circ 59.7'$$

$$a = 30.37903^\circ$$

Note: 1' arc in the terrestrial sphere = 1 nautical mile; $1^\circ = 60' = 60$ nautical miles

Therefore,

$$30.37903^\circ \cdot \frac{60 \text{ nautical miles}}{1^\circ} = \mathbf{1822.74 \text{ nautical miles}}$$



53. Tokyo is located at (139° E, 39°N) while Manila is at (121° E, 14° N). Find the distance between the two in nautical miles.

Solution:

You may refer to the solution in the previous problem.

Considering the formula:

$$\cos(90^\circ - \theta) = \sin \theta$$

To we can modify the cosine law of sides:

$$\cos a = \sin b \sin c + \cos b \cos c \cos A$$

$$\cos a = \sin 39^\circ \sin 14^\circ + \cos 39^\circ \cos 14^\circ \cos(139^\circ - 121^\circ)$$

$$a = 29.61^\circ$$

Therefore:

$$29.61^\circ \cdot \frac{60 \text{ nautical miles}}{1^\circ} = \mathbf{1776.65 \text{ nautical miles}}$$

PLANE ANALYTIC GEOMETRY

1. Find the slope of the line $3x - 2y + 3 = 0$.

A. $3/2$	C. $-3/2$
B. $2/3$	D. $-2/3$

2. Compute for the acute angle between the lines whose slopes are, $m_1 = 1/3$ and $m_2 = 2$.

A. 45°	C. 22.456°
B. 60°	D. 54.162°

3. What is the acute angle formed between the lines $4x - y + 3 = 0$ and $2x - 5y - 1 = 0$?

A. 45°	C. 22.456°
B. 60°	D. 54.162°

4. Find the distance between the points A (-2,3) and B (6,8).

A. 14.15	C. 6.40
B. 9.43	D. 13.60

5. Find the area of the triangle whose vertices are at points A (3,4), B (-2, 1) and C (5, -6).

A. 56	C. 28
B. 14	D. 26

6. Find the shortest distance from point (2,3) to the line $3x + 2y - 5 = 0$.

A. 2.94	C. 1.94
B. 3.44	D. 2.52

7. What is the distance between the lines: $5x - 2y + 3 = 0$ and $5x - 2y + 7 = 0$?

A. 0.743	C. 0.190
B. 1.346	D. 5.385

8. The curve $x^2 + y^2 - 4x + 2y - 4 = 0$ has its center at?

A. (2, 1)	C. (2, -1)
B. (-2, -1)	D. (-2, 1)

9. What is the radius of the curve $x^2 + y^2 - 6x - 8y - 11 = 0$?

A. 11	C. 1
B. 6	D. 3

10. Find the centroid of the triangle whose vertices are at points (3, 4), (6, -9) and (-6, 2).

A. (1, 1)	C. (-1, 1)
B. (1, -1)	D. (-1, -1)

11. The equation of the line passing through points (2, 4) and (5, 8) is?

A. $3x + 4y = -4$	C. $3x - 4y = 4$
B. $4x - 3y = -4$	D. $4x + 3y = 4$

12. Two lines $3x + 2y - 3 = 0$ and $Ax - 3y + 2 = 0$ are perpendicular to each other. Determine the value of A.
- A. 2
 B. 0
 C. -2
 D. -1
13. What is the equation of the line with slope equal to 3 and y-intercept of 5?
- A. $y = 5x + 3$
 B. $y = 5x - 3$
 C. $y = -3x - 5$
 D. $y = 3x + 5$
14. What is the equation of the line that passes through the origin and perpendicular to the line $3x - 4y + 3 = 0$?
- A. $3x + 4y = 0$
 B. $4x - 3y = 0$
 C. $4x - 3y = 0$
 D. $4x + 3y = 0$
15. What is the equation of the line that passes through the origin and parallel to the line $5x - 2y + 3 = 0$?
- A. $2x - 5y = 0$
 B. $5x + 2y = 0$
 C. $5x - 2y = 0$
 D. $2x + 5y = 0$
16. A circle passes through the point (5, 7) and has its center at (2, 3). Find its equation.
- A. $(x + 2)^2 + (y - 3)^2 = 25$
 B. $(x - 2)^2 + (y - 3)^2 = 25$
 C. $(x - 2)^2 + (y + 3)^2 = 25$
 D. $(x + 2)^2 + (y + 3)^2 = 25$
17. How far is the center of the circle $x^2 + y^2 - 6x - 8y - 11 = 0$ from the y-axis?
- A. 3
 B. 4
 C. 6
 D. 2
18. Find the vertex of the parabola whose equation is $y^2 - 2x - 4y + 2 = 0$.
- A. (1, -2)
 B. (-1, 2)
 C. (2, -1)
 D. (2, 1)
19. Determine the equation of the directrix of the parabola: $y^2 = 16x$.
- A. $x = 4$
 B. $y = -4$
 C. $x = -4$
 D. $y = 4$
20. A parabola has an equation of $x^2 - 4y - 2x + 8 = 0$. Find the length of the latus rectum.
- A. 1
 B. 16
 C. 4
 D. 8
21. A parabola has its vertex at (2, 4) and focus at (4, 4). Determine its equation.
- A. $(y + 4)^2 = 4(x - 2)$
 B. $(y + 4)^2 = 4(x + 2)$
 C. $(y - 4)^2 = 8(x - 2)$
 D. $(y - 4)^2 = 8(x - 2)$
22. The eccentricity of a parabola is always equal to?
- A. 0
 B. greater than 1
 C. less than 1
 D. 1

PLANE ANALYTIC GEOMETRY SOLUTIONS

1. Find the slope of the line $3x - 2y + 3 = 0$.

First Solution:

Transform the equation in the form $y = mx + b$:

$$\begin{aligned} 3x - 2y + 3 &= 0 \\ 2y &= 3x + 3 \\ y &= \frac{3}{2}x + \frac{3}{2} \\ \therefore m &= \frac{3}{2} \end{aligned}$$

Second solution:

The slope of the line in the form $Ax + By + C = 0$ is $m = -\frac{A}{B}$:

$$\begin{aligned} m &= \frac{-3}{-2} \\ m &= \frac{3}{2} \end{aligned}$$

2. Compute for the acute angle between the lines whose slopes are, $m_1 = 1/3$ and $m_2 = 2$.

Solution:

$$\begin{aligned} \tan \theta &= \frac{m_2 - m_1}{1 + m_1 m_2} \\ \tan \theta &= \frac{2 - \frac{1}{3}}{1 + (2) \left(\frac{1}{3}\right)} \\ \tan \theta &= 1 \\ \theta &= 45^\circ \end{aligned}$$

3. What is the acute angle formed between the lines $4x - y + 3 = 0$ and $2x - 5y - 1 = 0$?

Solution:

$$\begin{aligned} 4x - y + 3 &= 0 \rightarrow \text{equation 1} \\ 2x - 5y - 1 &= 0 \rightarrow \text{equation 2} \end{aligned}$$

$$\begin{aligned} m_1 &= -\frac{A_1}{B_1} = \frac{-4}{-1} = 4 \\ m_2 &= -\frac{A_2}{B_2} = \frac{-2}{-5} = \frac{2}{5} \end{aligned}$$

$$\begin{aligned} \tan \theta &= \frac{m_2 - m_1}{1 + m_1 m_2} \\ \tan \theta &= \frac{\frac{2}{5} - 4}{1 + (2) \left(\frac{2}{5}\right)} \\ \theta &= -54.162^\circ \text{ or } \theta = 54.162^\circ \end{aligned}$$

4. Find the distance between the points A (-2, 3) and B (6, 8).

Solution:

By distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(6 + 2)^2 + (8 - 3)^2}$$

$$\mathbf{d = \sqrt{89}}$$

5. Find the area of the triangle whose vertices are at points A (3, 4), B (-2, 1) and C (5, -6).

Solution:

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \begin{vmatrix} x_1 \\ y_1 \end{vmatrix} = \begin{vmatrix} 3 & -2 & 5 \\ 4 & 1 & -6 \end{vmatrix} \begin{vmatrix} 3 \\ 4 \end{vmatrix}$$

$$2A = |(x_1y_2 + x_2y_3 + x_3y_1) - (y_1x_2 + y_2x_3 + y_3x_1)|$$

$$2A = |[(3)(1) + (-2)(-6) + (5)(4)] - [(4)(-2) + (1)(5) + (-6)(3)]|$$

$$2A = 56$$

$$\mathbf{A = 28 \text{ sq. units}}$$

6. Find the shortest distance from point (2, 3) to the line $3x + 2y - 5 = 0$.

Solution:

The distance from a point (x_0, y_0) to a line $Ax + By + C = 0$ is:

$$d = \left| \frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}} \right|$$

$$d = \left| \frac{(3)(2) + (2)(3) - 5}{\sqrt{3^2 + 2^2}} \right|$$

$$\mathbf{d = 1.94}$$

7. What is the distance between the lines: $5x - 2y + 3 = 0$ and $5x - 2y + 7 = 0$?

Solution:

Distance between two parallel lines:

$$d = \left| \frac{C_2 - C_1}{\sqrt{A^2 + B^2}} \right|$$

$$d = \left| \frac{7 - 3}{\sqrt{5^2 + (-2)^2}} \right|$$

$$\mathbf{d = 0.743}$$

8. The curve $x^2 + y^2 - 4x + 2y - 4 = 0$ has its center at?

First Solution:

$$x^2 - 4x + \left(\frac{4}{2}\right)^2 + y^2 + 2y + \left(\frac{2}{2}\right)^2 = 4 + \left(\frac{4}{2}\right)^2 + \left(\frac{2}{2}\right)^2$$

$$(x - 2)^2 + (y + 1)^2 = 9$$

$$\therefore \mathbf{C(2, -1)}$$

Second solution:

Type:

$$-\frac{B}{2A}$$

Press CALC:

Solve for h:

$$B = ? B = -4$$

$$A = ? A = 1$$

$$h = 2$$

Solve for k:

$$B = ? B = 2$$

$$A = ? A = 1$$

$$k = -1$$

$$\therefore \mathbf{C(2, -1)}$$

9. What is the radius of the curve $x^2 + y^2 - 6x - 8y - 11 = 0$?

Solution:

By completing the square:

$$\begin{aligned} x^2 - 6x + \left(\frac{6}{2}\right)^2 + y^2 - 8y + \left(\frac{8}{2}\right)^2 &= 11 + \left(\frac{6}{2}\right)^2 + \left(\frac{8}{2}\right)^2 \\ (x - 3)^2 + (y - 4)^2 &= 6^2 \\ \therefore \mathbf{r = 6} \end{aligned}$$

10. Find the centroid of the triangle whose vertices are at points (3, 4), (6, -9) and (-6, 2).

Solution:

$$\bar{x} = x_{\text{mean}} = \frac{x_1 + x_2 + x_3}{3}$$

$$\bar{x} = \frac{3 + 6 + (-6)}{3}$$

$$\bar{x} = 1$$

$$\bar{y} = y_{\text{mean}} = \frac{y_1 + y_2 + y_3}{3}$$

$$\bar{y} = \frac{4 + (-9) + 2}{3}$$

$$\bar{y} = -1$$

$$\therefore \mathbf{C(1, -1)}$$

11. The equation of the line passing through points (2,4) and (5,8) is?

Solution:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 4}{5 - 2} = \frac{4}{3}$$

$$\frac{4}{3} = \frac{y - y_1}{x - x_1}$$

$$\frac{4}{3} = \frac{y - 4}{x - 2}$$

$$4x - 8 = 3y - 12$$

$$\mathbf{4x - 3y + 4 = 0}$$

12. Two lines $3x + 2y - 3 = 0$ and $Ax - 3y + 2 = 0$ are perpendicular to each other. Determine the value of A.

Solution:

$$3x + 2y - 3 = 0 \rightarrow \text{equation 1}$$

$$y = -\frac{3}{2}x + \frac{3}{2}$$

$$m_1 = -\frac{3}{2}$$

$$Ax - 3y + 2 = 0 \rightarrow \text{equation 2}$$

$$y = \frac{A}{3}x + \frac{2}{3}$$

$$m_2 = \frac{A}{3}$$

$$m_2 = -\frac{1}{m_1}$$

$$\frac{A}{3} = -\frac{1}{-\frac{3}{2}}$$

$$\mathbf{A = 2}$$

13. What is the equation of the line with slope equal to 3 and y-intercept of 5?

Solution:

Equation of a line:

$$y = mx + b$$

$$\mathbf{y = 3x + 5}$$

14. What is the equation of the line that passes through the origin and perpendicular to the line $3x - 4y + 3 = 0$?

Solution:

$$3x - 4y + 3 = 0$$

$$y = \frac{3}{4}x + \frac{3}{4}$$

$$m_1 = \frac{3}{4}$$

$$m_2 = -\frac{1}{m_1} = -\frac{1}{\frac{3}{4}} = -\frac{4}{3}$$

The slope of the line is $-4/3$ and passes through the origin $(0,0)$.

$$-\frac{4}{3} = \frac{y-0}{x-0}$$

$$-4x = 3y$$

$$\mathbf{4x + 3y = 0}$$

15. What is the equation of the line that passes through the origin and parallel to the line $5x - 2y + 3 = 0$?

Solution:

$$5x - 2y + 3 = 0$$

$$y = \frac{5}{2}x + \frac{3}{2}$$

$$\therefore m = \frac{5}{2}$$

The line is parallel to line 1. Thus, its slope is also equal to $5/2$. It passes through the origin.

$$\frac{5}{2} = \frac{y-0}{x-0}$$

$$\mathbf{5x - 2y = 0}$$

16. A circle passes through the point $(5, 7)$ and has its center at $(2, 3)$. Find its equation.

Solution:

Standard equation of the circle:

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - 2)^2 + (y - 3)^2 = r^2$$

It passes through $(5,7)$. Thus:

$$(5 - 2)^2 + (7 - 3)^2 = r^2$$

$$r = 5$$

$$\therefore \mathbf{(x - 2)^2 + (y - 3)^2 = 25}$$

17. How far is the center of the circle $x^2 + y^2 - 6x - 8y - 11 = 0$ from the y-axis?

Solution:

By completing the square:

$$x^2 - 6x + \left(\frac{6}{2}\right)^2 + y^2 - 8y + \left(\frac{8}{2}\right)^2 = 11 + \left(\frac{6}{2}\right)^2 + \left(\frac{8}{2}\right)^2$$

$$\mathbf{(x - 3)^2 + (y - 4)^2 = 6^2}$$

The distance from the center (h, k) of a circle from the y-axis is h . Therefore:

$$\mathbf{h = 3}$$

18. Find the vertex of the parabola whose equation is $y^2 - 2x - 4y + 2 = 0$.

Solution:

$$\begin{aligned} y^2 - 4y &= 2x - 2 \\ y^2 - 4y + \left(\frac{4}{2}\right)^2 &= 2x - 2 + \left(\frac{4}{2}\right)^2 \\ (y - 2)^2 &= 2x + 2 \\ (y - 2)^2 &= 2(x + 1) \end{aligned}$$

Therefore, the vertex is at:

$$\mathbf{V(-1, 2)}$$

19. Determine the equation of the directrix of the parabola: $y^2 = 16x$.

Solution:

$$y^2 = 16x$$

The curve is a parabola that concaves to the right having a latus rectum equal to 16. Its vertex is at origin.

$$\begin{aligned} \text{LR} &= 16 = 4a \\ a &= 4 \end{aligned}$$

The distance from the vertex to the directrix is $a = 4$ to the left. Therefore, the equation of the directrix is:

$$\mathbf{x = -4}$$

20. A parabola has an equation of $x^2 - 4y - 2x + 8 = 0$. Find the length of the latus rectum.

Solution:

$$\begin{aligned} x^2 - 4y - 2x + 8 &= 0 \\ x^2 - 2x &= 4y - 8 \\ x^2 - 2x + \left(\frac{2}{2}\right)^2 &= 4y - 8 + \left(\frac{2}{2}\right)^2 \\ (x - 1)^2 &= 4y - 7 \\ (x - 1)^2 &= 4\left(y - \frac{7}{4}\right) \\ \therefore \text{LR} &= 4 \end{aligned}$$

21. A parabola has its vertex at (2,4) and focus at (4,4). Determine its equation.

Solution:

$$\text{LR} = 4a = 4(2) = 8$$

The equation is:

$$\begin{aligned} (y - k)^2 &= \text{LR}(x - h) \\ (\mathbf{y - 4})^2 &= \mathbf{8(x - 2)} \end{aligned}$$

22. The eccentricity of a parabola is always equal to?

Solution:

$$\begin{aligned} e &= 1 \text{ (parabola)} \\ e &< 1 \text{ (ellipse)} \\ e &> 1 \text{ (hyperbola)} \end{aligned}$$

The eccentricity of a parabola is always **equal to 1**.

23. What is the eccentricity of an ellipse whose length of major and minor axes equal to 5 and 4, respectively, with center at origin?

Solution:

$$a = 5; b = 4$$

$$a^2 = b^2 + c^2$$

$$5^2 = 4^2 + c^2$$

$$c = 3$$

$$e = \frac{c}{a}$$

$$e = \frac{3}{5}$$

24. Find the second eccentricity of the curve $9x^2 + 4y^2 - 36x - 8y + 4 = 0$.

Solution:

$$9x^2 + 4y^2 - 36x - 8y = -4$$

$$9(x^2 - 4x + 4) + 4(y^2 - 2y + 1) = -4 + 9(4) + 4(1)$$

$$9(x - 2)^2 + 4(y - 1)^2 = 36$$

$$\frac{(x - 2)^2}{2^2} + \frac{(y - 1)^2}{3^2} = 1$$

$$a = 3; b = 2$$

$$a^2 = b^2 + c^2$$

$$3^2 = 2^2 + c^2$$

$$c = \sqrt{5}$$

The second eccentricity, e' is:

$$e' = \frac{c}{b} = \frac{\sqrt{5}}{2}$$

25. Convert $r = 3\cos\theta$ into Cartesian coordinates.

Solution:

$$r^2 = x^2 + y^2$$

$$\cos\theta = \frac{x}{r}$$

Therefore:

$$\sqrt{x^2 + y^2} = \frac{3x}{\sqrt{x^2 + y^2}}$$

$$x^2 + y^2 = 3x$$

$$x^2 + y^2 - 3x = 0$$

26. Convert $x^2 + y^2 - 2x + 4y = 0$ into polar form.

Solution:

$$x^2 + y^2 - 2x + 4y = 0$$

$$r^2 - 2(r\cos\theta) + 4(r\sin\theta) = 0$$

$$r - 2\cos\theta + 4\sin\theta = 0$$

27. Determine the polar coordinates of a point having a Cartesian coordinate of (3,4).

First Solution:

$$\begin{aligned}(3,4) &\rightarrow r\angle\theta \\ r &= \sqrt{3^2 + 4^2} = 5 \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \\ \theta &= \tan^{-1}\left(\frac{4}{3}\right) \\ \theta &= 53.13^\circ \\ \therefore &\mathbf{5\angle 53.13^\circ}\end{aligned}$$

Second solution:

Type:

$$\text{Pol}(3,4)$$

Then press =

$$\mathbf{r = 5, \theta = 53.13^\circ}$$

28. Determine the Cartesian coordinate of a point having a polar coordinate of (12, 30°).

First Solution:

$$\begin{aligned}x &= r \cos \theta \\ x &= 12 \cos 30^\circ \\ x &= 10.392\end{aligned}$$

$$\begin{aligned}y &= r \sin \theta \\ y &= 10 \sin 30^\circ \\ y &= 6\end{aligned}$$

Therefore:

$$\mathbf{(10.392, 6)}$$

Second solution:

Type:

$$\begin{aligned}\text{Rec}(12,30^\circ) \\ \mathbf{x = 10.392, y = 6}\end{aligned}$$

29. Determine the flatness of the ellipse $9x^2 + 4y^2 - 36x - 8y + 4 = 0$.

Solution:

$$\begin{aligned}9x^2 + 4y^2 - 36x - 8y &= -4 \\ 9(x^2 - 4x + 4) + 4(y^2 - 2y + 1) &= -4 + 9(4) + 4(1) \\ 9(x - 2)^2 + 4(y - 1)^2 &= 36 \\ \frac{(x - 2)^2}{2^2} + \frac{(y - 1)^2}{3^2} &= 1 \\ a &= 3, b = 2\end{aligned}$$

Ellipse flatness, f:

$$\begin{aligned}f &= \frac{a - b}{b} \\ f &= \frac{3 - 2}{2} \\ \mathbf{f} &= \mathbf{\frac{1}{2}}\end{aligned}$$

30. Find the distance between points (2, 3, 5) and (-1, 4, -2) in space.

Solution:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$d = \sqrt{(-1 - 2)^2 + (4 - 3)^2 + (-2 - 5)^2}$$

$$d = \sqrt{59}$$

31. What is the radius of a sphere that passes through (1, 3, 6) and has its center at the origin?

Solution:

The radius of the sphere is equal to the distance from its center to the point on the sphere.

$$r = \sqrt{(1 - 0)^2 + (3 - 0)^2 + (6 - 0)^2}$$

$$r = \sqrt{46}$$

32. Determine the point of intersection of the planes: $4x - 3y + 2z = 7$, $x + 2y - z = 4$ and $5x + y + 2z = 21$.

Solution:

At the point of intersection:

$$x_1 = x_2 = x_3$$

$$y_1 = y_2 = y_3$$

$$z_1 = z_2 = z_3$$

Therefore, we have to solve the three equations simultaneously:

Go to MODE 5-2:

$$\begin{vmatrix} 4 & -3 & 2 & 7 \\ 1 & 2 & -1 & 4 \\ 5 & 1 & 2 & 21 \end{vmatrix}$$

$$x = 2; y = 3; z = 4$$

$$\therefore (2, 3, 4)$$

33. Determine the equation of the asymptote of the hyperbola $\frac{x^2}{49} - \frac{y^2}{36} = 1$.

Solution:

The asymptotes of the hyperbola

$$\frac{x^2}{7^2} - \frac{y^2}{6^2} = 1$$

passes through the origin and have slopes $m_1 = \frac{6}{7}$ and $m_2 = -\frac{6}{7}$. Thus:

$$\pm \frac{6}{7} = \frac{y - 0}{x - 0}$$

$$6x \pm 7y = 0$$

34. Find the distance between foci of the curve $9x^2 + 4y^2 - 36x - 8y + 4 = 0$.

Solution:

$$\begin{aligned}
 9x^2 + 4y^2 - 36x - 8y &= -4x^2 + 4y^2 - 36x - 8y = -4 \\
 9(x - 4x + 4) + 4(y^2 - 2y + 1) &= -4 + 9(4) + 4(1) \\
 9(x - 2)^2 + 4(y - 1)^2 &= 36 \\
 \frac{(x - 2)^2}{2^2} + \frac{(y - 1)^2}{3^2} &= 1 \\
 a &= 3; b = 2 \\
 a^2 &= b^2 + c^2 \\
 3^2 &= 2^2 + c^2 \\
 c &= \sqrt{5}
 \end{aligned}$$

Distance between foci is $2c$:

$$2c = 2\sqrt{5}$$

35. The eccentricity of the curve $9x^2 - 4y^2 = 36$ is?

Solution:

$$\begin{aligned}
 9x^2 - 4y^2 &= 36 \\
 \frac{x^2}{2^2} - \frac{y^2}{3^2} &= 1 \\
 a &= 2; b = 3 \\
 c^2 &= a^2 + b^2 \\
 c^2 &= 2^2 + 3^2 \\
 c &= \sqrt{13}
 \end{aligned}$$

The eccentricity, e is:

$$e = \frac{c}{a} = \frac{\sqrt{13}}{2}$$

36. Find the locus of points such that the distance from $(2, 3)$ to any point on the curve is twice the distance from the line $x = 3$ to that point on the curve.

Solution:

Let $P(x, y)$ be any point on the curve:

$$\begin{aligned}
 \sqrt{(x - 2)^2 + (y - 3)^2} &= 2\sqrt{(x - 3)^2} \\
 (x - 2)^2 + (y - 3)^2 &= 4(x - 3)^2 \\
 x^2 - 4x + 4 + y^2 - 6y + 9 &= 4x^2 - 24x + 36 \\
 \mathbf{3x^2 - y^2 - 20x + 6y + 23 = 0}
 \end{aligned}$$

37. Find the new equation of the curve $5x - 2y = 2$ if the origin is translated to the point $(2, -3)$.

Solution:

The origin is translated to $(2, -3)$. So, we put $h = 2$ and $k = -3$. The translation formulas are:

$$\begin{aligned}
 x &= x' + h \\
 y &= y' + k
 \end{aligned}$$

$$\begin{aligned}
 5x - 2y &= 2 \\
 5(x' + 2) - 2(y' - 3) &= 2 \\
 5x' + 10 - 2y' + 6 &= 2 \\
 \mathbf{5x' - 2y' + 14 = 0}
 \end{aligned}$$

38. Determine the new equation of the parabola $y^2 - 6x + 4y + 22 = 0$ if the origin is translated to its vertex.

Solution:

$$\begin{aligned} y^2 - 6x + 4y + 22 &= 0 \\ y^2 + 4y + \left(\frac{4}{2}\right)^2 &= 6x - 22 + \left(\frac{4}{2}\right)^2 \\ (y + 2)^2 &= 6x - 18 \\ (y + 2)^2 &= 6(x - 3) \\ \therefore V(3, -2) \end{aligned}$$

So, we put $h = 3$ and $k = -2$:

$$\begin{aligned} y^2 - 6x + 4y + 22 &= 0 \\ (y' - 2)^2 - 6(x' + 3) + 4(y' - 2) + 22 &= 0 \\ (y')^2 - 4y' + 4 - 6x' - 18 + 4y' - 8 + 22 &= 0 \\ y'^2 - 6x' &= 0 \\ \mathbf{y'^2 = 6x'} \end{aligned}$$

39. Find the new equation of the curve $\frac{(x-4)^2}{36} + \frac{(y-3)^2}{25} = 1$ if the origin is translated to the center of the ellipse.

Solution:

$$C(4,3)$$

So, we put $h = 4$ and $k = 3$:

$$\begin{aligned} \frac{(x - 4)^2}{36} + \frac{(y - 3)^2}{25} &= 1 \\ \frac{(x' + 4 - 4)^2}{36} + \frac{(y' + 3 - 3)^2}{25} &= 1 \\ \mathbf{25(x')^2 + 36(y')^2 = 900} \end{aligned}$$

40. Find the point to which the origin must be translated in order that the transformed equation of the curve $4x^2 + 4y^2 - 8x + 4y + 1 = 0$ will have no first-degree term.

Solution:

$$\begin{aligned} x &= x' + h \\ y &= y' + k \end{aligned}$$

$$\begin{aligned} 4x^2 + 4y^2 - 8x + 4y + 1 &= 0 \\ 4(x' + h)^2 + 4(y' + k)^2 - 8(x' + h) + 4(y' + k) + 1 &= 0 \\ 4(x'^2 + 2x'h + h^2) + 4(y'^2 + 2y'k + k^2) - 8x' - 8h + 4y' + 4k + 1 &= 0 \\ 4x'^2 + 8x'h + 4h^2 + 4y'^2 + 8y'k + 4k^2 - 8x' - 8h + 4y' + 4k + 1 &= 0 \\ 4x'^2 + (8h - 8)x' + 4y'^2 + (8k + 4)y' + 4h^2 + 4k^2 - 8h + 4k + 1 &= 0 \end{aligned}$$

We want to eliminate the first-degree terms in the new equation. Thus, we equate the coefficient of x' and y' to zero.

$$\begin{aligned} (8h - 8) &= 0; h = 1 \\ (8k + 4) &= 0; k = -\frac{1}{2} \end{aligned}$$

The origin must be translated to $\left(1, -\frac{1}{2}\right)$, the center of the circle.

41. Determine the new equation of the curve $xy = 4$ when the axes are rotated 45° .

Solution:

The rotation formulas are:

$$\begin{aligned}x &= x' \cos \theta - y' \sin \theta \\y &= x' \sin \theta + y' \cos \theta\end{aligned}$$

$$\begin{aligned}xy &= 4 \\(x' \cos \theta - y' \sin \theta)(x' \sin \theta + y' \cos \theta) &= 4 \\x'^2 \sin \theta \cos \theta + x'y' \cos^2 \theta - x'y' \sin^2 \theta - y'^2 \sin \theta \cos \theta &= 4 \\x'^2 \sin \theta \cos \theta + x'y'(\cos^2 \theta - \sin^2 \theta) - y'^2 \sin \theta \cos \theta &= 4 \\x'^2 + x'y' \cos 2\theta - y'^2 \sin \theta \cos \theta &= 4\end{aligned}$$

Substitute $\theta = 45^\circ$:

$$\begin{aligned}x'^2 \left(\frac{1}{2}\right) + 0 - y'^2 \left(\frac{1}{2}\right) &= 4 \\x'^2 - y'^2 &= 8\end{aligned}$$

42. Find the new equation of the curve $x^2 = 16y$ if the axes are rotated 30° .

Solution:

$$\begin{aligned}x^2 &= 16y; \theta = 30^\circ \\(x' \cos \theta - y' \sin \theta)^2 &= 16(x' \sin \theta + y' \cos \theta) \\x'^2 \cos^2 \theta - 2x'y' \sin \theta \cos \theta + y'^2 \sin^2 \theta &= 16x' \sin \theta + 16y' \cos \theta \\\frac{3}{4}x'^2 - 2x'y' \left(\frac{\sqrt{3}}{4}\right) + \frac{1}{4}y'^2 &= 8x' + 8\sqrt{3}y' \\3x'^2 - 2\sqrt{3}x'y' + y'^2 &= 32x' + 32\sqrt{3}y'\end{aligned}$$

43. What is the new equation of the line $3x + 2y = 1$ if the axes are rotated 60° ?

Solution:

$$\begin{aligned}3x + 2y &= 1; \theta = 60^\circ \\3(x' \cos \theta - y' \sin \theta) + 2(x' \sin \theta + y' \cos \theta) &= 1 \\3x' \cos \theta - 3y' \sin \theta + 2x' \sin \theta + 2y' \cos \theta &= 1 \\(3 \cos \theta + 2 \sin \theta)x' + (2 \cos \theta - 3 \sin \theta)y' &= 1 \\\frac{3 + 2\sqrt{3}}{2}x' + \frac{2 - 3\sqrt{3}}{2}y' &= 1 \\(3 + 2\sqrt{3})x' + (2 - 3\sqrt{3})y' &= 2\end{aligned}$$

44. Transform the equation $x^2 - y^2 = 25$ by rotating the axes through 45° .

Solution:

$$\begin{aligned}x^2 - y^2 &= 25; \theta = 45^\circ \\\cos 45^\circ &= \frac{\sqrt{2}}{2}; \sin \theta = \frac{\sqrt{2}}{2} \\(x' \cos \theta - y' \sin \theta)^2 - (x' \sin \theta + y' \cos \theta)^2 &= 25 \\\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)^2 - \left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2 &= 25 \\(x'^2 - 2x'y' - y'^2) - (x'^2 + 2x'y' + y'^2) &= 50 \\2x'y' + 25 &= 0\end{aligned}$$

45. Find the acute angle of rotation such that the transformed equation of $7x^2 + \sqrt{3}xy + 6y^2 = 16$ will have no $x'y'$ term.

Solution:

The general second-degree equation is:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

The $x'y'$ term will only vanish if its coefficient is zero. By rotation of axes, θ may be determine using:

$$\tan 2\theta = \frac{B}{A - C}; \text{ if } A \neq C$$

$$\tan 2\theta = \frac{\sqrt{3}}{1 - 6}$$

$$2\theta = 60^\circ; \theta = 30^\circ$$

46. Determine the angle of rotation such that the transformed equation of $xy = 1$ will have no $x'y'$ term.

Solution:

$$(x' \cos \theta - y' \sin \theta)(x' \sin \theta + y' \cos \theta) = 1$$

$$x'^2 \sin \theta \cos \theta + x'y' \cos^2 \theta - x'y' \sin^2 \theta - y'^2 \sin \theta \cos \theta = 1$$

$$(x')^2 \sin \theta \cos \theta + (\cos^2 \theta - \sin^2 \theta) - (y')^2 \sin \theta \cos \theta = 1$$

To eliminate $x'y'$ term, set the coefficient to zero:

$$\cos^2 \theta - \sin^2 \theta = 0$$

$$\cos 2\theta = 0$$

$$2\theta = 90^\circ$$

$$\theta = 45^\circ$$

47. Determine the center of the sphere $x^2 + y^2 + z^2 + 8x + 6y + 10z = 0$.

Solution:

$$x^2 + y^2 + z^2 + 8x + 6y + 10z = 0$$

$$x^2 + 8x + \left(\frac{8}{2}\right)^2 + y^2 + 6y + \left(\frac{6}{2}\right)^2 + z^2 + 10z + \left(\frac{10}{2}\right)^2$$

$$= \left(\frac{8}{2}\right)^2 + \left(\frac{6}{2}\right)^2 + \left(\frac{10}{2}\right)^2$$

$$(x + 4)^2 + (y + 3)^2 + (z + 5)^2 = 50$$

$$\therefore \mathbf{C (-4, -3, -5)}$$

48. Find the radius of the sphere $x^2 + y^2 + z^2 + 2x - 6y - 12z = 0$.

Solution:

$$x^2 + y^2 + z^2 + 2x - 6y - 12z = 0$$

$$x^2 + 2x + \left(\frac{2}{2}\right)^2 + y^2 - 6y + \left(\frac{6}{2}\right)^2 + z^2 - 12z + \left(\frac{12}{2}\right)^2$$

$$= \left(\frac{2}{2}\right)^2 + \left(\frac{6}{2}\right)^2 + \left(\frac{12}{2}\right)^2$$

$$(x + 1)^2 + (y - 3)^2 + (z - 6)^2 = 46$$

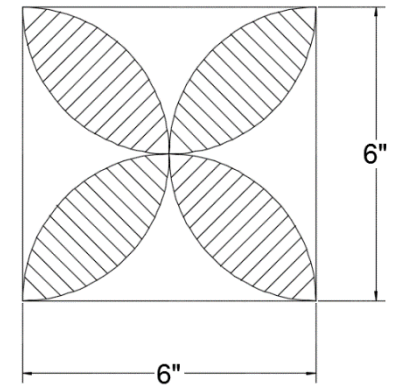
$$R^2 = 46$$

$$\mathbf{R = \sqrt{46}}$$

13. In a circle with diameter of 20 cm, a regular five-pointed star touching its circumference is inscribed. Find the area of the star.

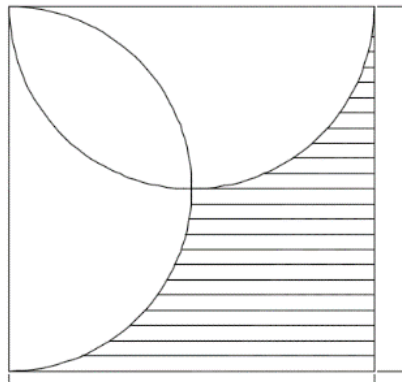
- A. 159.25 cm^2 C. 191.03 cm^2
 B. 112.26 cm^2 D. 203.67 cm^2

14. If the area of the square shown in the figure is 36 square inches, find the area of the inscribed four-leaf figure. (See figure)



- A. 20.55 in^2 C. 18.76 in^2
 B. 16.74 in^2 D. 23.66 in^2

15. Two semicircles are inscribed in a square with side 8 cm as shown. Find the area of the shaded region. (See figure)



- A. 25.78 cm^2 C. 22.87 cm^2
 B. 19.56 cm^2 D. 28.55 cm^2

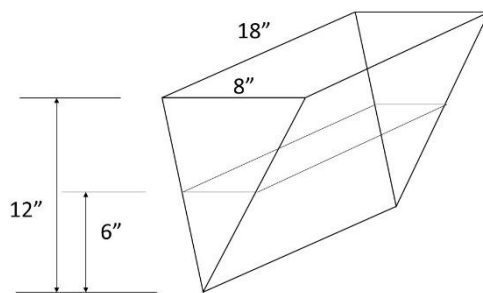
16. The corresponding edges of two regular tetrahedrons are 1 cm and 3 cm. If the sum of the weights of the two tetrahedrons is 100 grams and both solids are made up of the same material, find the weight of the bigger solid.

- A. 96.43 grams C. 79.35 grams
 B. 88.02 grams D. 93.25 grams

17. Find the volume of a regular tetrahedron whose altitude measures 10 cm.

- A. 344.95 cm^3 C. 433.49 cm^3
 B. 503.63 cm^3 D. 216.51 cm^3

18. A container in the form of a triangular prism as shown in the figure below contains water at a depth of 6 inches. Find the volume of water. (See figure)



- A. 216 in^3
- B. 432 in^3
- C. 108 in^3
- D. 324 in^3

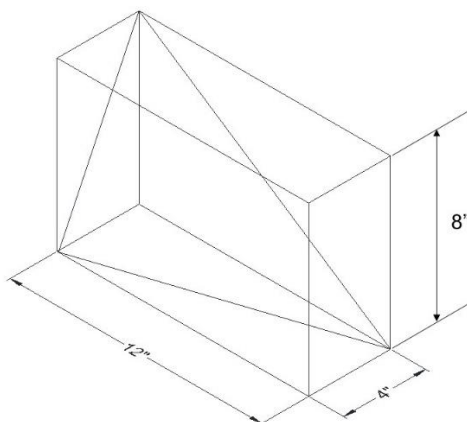
19. A flower vase, in the form of a hexagonal prism, is to be filled with 512 cubic inches of water. Find the height of the water if the wet portion of the flower vase and its volume are numerically equal.

- A. 1.22 inches
- B. 2.38 inches
- C. 34.86 inches
- D. 12.70 inches

20. In a regular hexagonal prism, the ratio of an edge of the base to the height of the prism is 1:5. Find the volume of the prism if the lateral area is 270 cm^2 .

- A. 350.74 cm^3
- B. 233.83 cm^3
- C. 175.37 cm^3
- D. 210.44 cm^3

21. Find the area of the triangular section of the rectangular solid shown in the figure. (See figure)



- A. 44.80 in^2
- B. 33.60 in^2
- C. 56.00 in^2
- D. 42.00 in^2

22. Find the edge of a cube if its surface area is numerically equal to its volume.

- A. 6 units
- B. 4 units
- C. 3 units
- D. 5 units

23. The base of a rectangular solid is 4 ft. long and 3 ft. wide. Find the volume of the solid if its diagonal is $\sqrt{41}$ ft.

- A. 72.00 ft^3
- B. 80.00 ft^3
- C. 48.00 ft^3
- D. 43.20 ft^3

24. The size of a brick is 25 cm x 12 cm x 55 cm and its density is $1.8 \times 10^3 \text{ kg/m}^3$. How many trips must a 3-ton dumper carry out to transport 15,000 bricks to a construction site? (Hint: 1 metric ton is equivalent to 1,000 kg.)

- A. 149 trips
- B. 155 trips
- C. 160 trips
- D. 190 trips

33. The base edges of a frustum of a regular pentagonal pyramid are 4 in and 8 in, and its altitude is 10 in. Find the volume of the frustum.
- A. 788.24 in^3 C. 789.52 in^3
B. 555.55 in^3 D. 642.31 in^3
34. The volume of a truncated right triangular prism is 72 ft^3 . The lengths of the two perpendicular edges of the base are 3 ft. and 4 ft., and two of the lateral edges measure 14 ft. and 12 ft. Find the measure of the third lateral edge.
- A. 15.22 ft C. 16.54 ft.
B. 10.00 ft. D. 8.60 ft.
35. In a truncated right circular cylinder, the top plane makes an angle of 60° with the horizontal, and the shortest and longest elements are 4 and 10 units, respectively. Find the volume of the solid.
- A. 21π cubic units C. 42π cubic units
B. $63/4 \pi$ cubic units D. $231/8 \pi$ cubic units
36. Suppose two cylindrical tubes with the same diameter intersect at right angles. If the diameter of each tube is 6 cm, find the volume of the common part between the tubes.
- A. $36 \pi \text{ cm}^3$ C. 144 cm^3
B. $72 \pi \text{ cm}^3$ D. 108 cm^3
37. The plane of a small circle in a sphere of diameter 18 inches is 6 inches from the center of the sphere. Find the area of the small circle.
- A. $54 \pi \text{ in}^2$ C. $72 \pi \text{ in}^2$
B. $10 \pi \text{ in}^2$ D. $45 \pi \text{ cm}^2$
38. A plane is tangent to a sphere at point A. A point B on this plane is 10 cm from the surface of the sphere and is 16 cm from point A. Find the surface area of the sphere.
- A. 674.78 cm^2 C. 764.54 cm^2
B. 746.34 cm^2 D. 874.36 cm^2
39. The base of a spherical pyramid is a bi-rectangular triangle whose third angle is 23° . If the diameter of the sphere is 90 cm, find the volume of the spherical pyramid.
- A. 12193 cm^3 C. 270.956 cm^3
B. 36579 cm^3 D. 1590.39 cm^3

SOLID MENSURATION SOLUTIONS

1. The number of diagonals of a regular polygon is 35. Find the area of the polygon if its apothem measures 10 centimeters.

Solution:

$$n = \frac{n(n - 3)}{2}$$

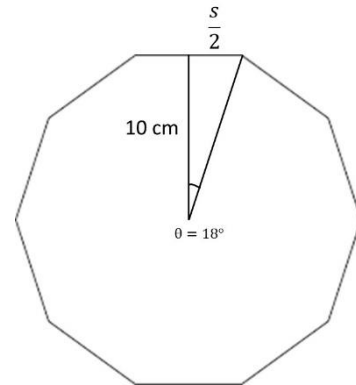
$$35 = \frac{n(n - 3)}{2}$$

$$n = 10$$

From the figure,

$$\tan 18^\circ = \frac{s/2}{10}$$

$$s = 6.50 \text{ cm}$$



From the formula of the area,

$$A = \frac{1}{2} Pa$$

$$A = \frac{1}{2} nsa$$

$$A = \frac{1}{2} (10)(6.50)(10)$$

$$A = 324.92 \text{ cm}^2$$

2. Find the number of sides of each of the two polygons if the total number of sides of the polygons is 15, and the sum of the number of diagonals of the polygon is 36.

Solution:

$$n_1 + n_2 = 15 \rightarrow \text{equation 1}$$

$$d_1 + d_2 = 36 \rightarrow \text{equation 2}$$

From equation 1:

$$n_2 = 15 - n_1$$

From equation 2:

$$\frac{n_1(n_1 - 3)}{2} + \frac{n_2(n_2 - 3)}{2} = 36$$

$$n_1(n_1 - 3) + n_2(n_2 - 3) = 72 \rightarrow \text{equation 3}$$

Substitute equation 1 to equation 3:

$$n_1(n_1 - 3) + (15 - n_1)[(15 - n_1) - 3] = 72$$

Solving for n_1 :

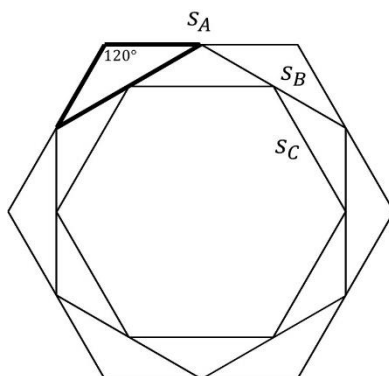
$$n_1 = 9$$

$$n_2 = 6$$

3. A regular hexagon A has the midpoints of its edges joined to form a smaller hexagon B. This process is repeated by joining the midpoints of the edges of hexagon B to get a third hexagon C. What is the ratio of the area of hexagon C to the area of hexagon A?

Solution:

Referring to the figure:



$$\text{interior angles} = \frac{(n - 2)}{n} 180^\circ$$

$$\text{interior angles} = \frac{(6 - 2)}{6} (180^\circ)$$

$$\text{interior angles} = 120^\circ$$

By cosine law,

$$(s_B)^2 = \left(\frac{s_A}{2}\right)^2 + \left(\frac{s_A}{2}\right)^2 - 2\left(\frac{s_A}{2}\right)\left(\frac{s_A}{2}\right)\cos 120^\circ$$

$$(s_B)^2 = \frac{3}{4}(s_A)^2$$

$$s_B = \frac{\sqrt{3}}{2}s_A$$

Applying the same manner to hexagon B and hexagon C:

$$s_C = \frac{\sqrt{3}}{2}s_B$$

Applying principles of similarity:

$$\frac{A_A}{A_C} = \left(\frac{s_A}{s_C}\right)^2$$

Solve s_C in terms of s_A :

$$s_C = \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}s_A\right)$$

$$s_C = \frac{3}{4}s_A$$

Substituting:

$$\frac{A_A}{A_C} = \left(\frac{s_A}{\frac{3}{4}s_A}\right)^2$$

$$\frac{A_A}{A_C} = \frac{16}{9}$$

$$\frac{A_C}{A_A} = \frac{9}{16}$$

4. In a right triangle, the bisector of the right angle divides the hypotenuse in the ratio of 3 is to 5. Determine the measures of the acute angles of the triangle.

Solution:

Referring to the figure:

$$CD:AD = 3:5 \text{ or}$$

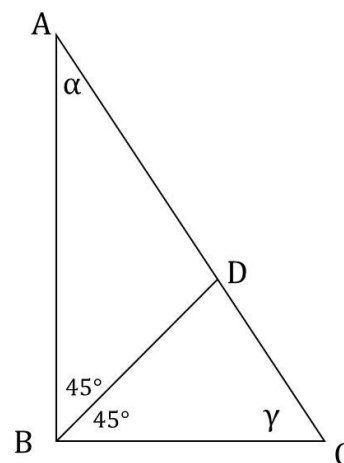
$$\frac{CD}{AD} = \frac{3}{5} \rightarrow AD = \frac{5}{3}CD$$

Isolate $\triangle BDC$:

From the figure:

By sine law:

$$\frac{CD}{\sin 45^\circ} = \frac{BD}{\sin \gamma}$$



Solving for BD:

$$BD = \sqrt{2}CD \sin \gamma \rightarrow \text{equation 1}$$

Isolate $\triangle ABD$:

From the figure:

By sine law:

$$\frac{BD}{\sin \gamma} = \frac{AD}{\sin 45^\circ} \text{ but } AD = \frac{5}{3}CD$$

$$\frac{BD}{\sin \alpha} = \frac{5/3 CD}{\sin 45^\circ}$$

Simplifying and solve for BD:

$$BD = \frac{5\sqrt{2}}{3}CD \sin \alpha \rightarrow \text{equation 2}$$

Equate equation 1 and equation 2:

$$\sqrt{2}CD \sin \gamma = \frac{5\sqrt{2}}{3}CD \sin \alpha$$

Simplifying:

$$\sin \gamma = \frac{5}{3} \sin \alpha$$

But,

$$\alpha + \gamma = 90^\circ \rightarrow \gamma = 90^\circ - \alpha$$

Substituting,

$$\sin(90^\circ - \alpha) = \frac{5}{3} \sin \alpha$$

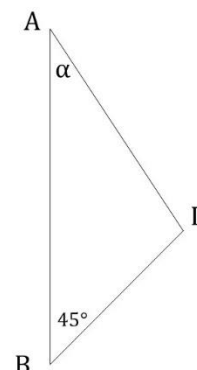
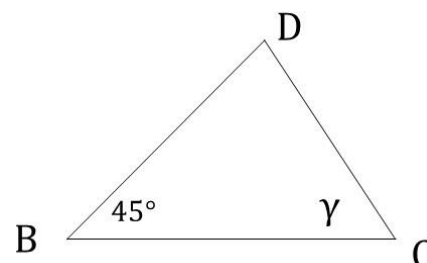
Applying trigonometric identities,

$$\cos \alpha = \frac{5}{3} \sin \alpha$$

$$\tan \alpha = \frac{3}{5}$$

$$\alpha = 30.96^\circ$$

$$\gamma = 59.04^\circ$$



5. In an acute triangle ABC, an altitude AD is drawn. Find the area of triangle ABC if AB = 15 inches, AC = 18 inches, and BD = 10 inches.

Solution:

Referring to the figure:

$$\sin \alpha = \frac{10}{15}$$

$$\alpha = 41.81^\circ$$

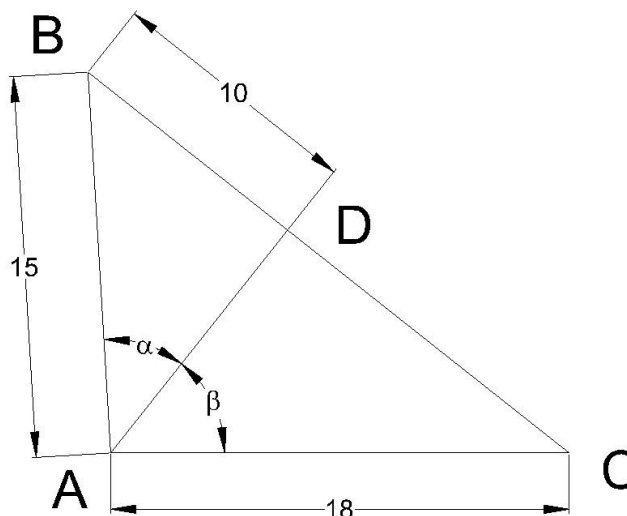
By Pythagorean Theorem:

$$(AD)^2 + 10^2 = 15^2$$

$$AD = 5\sqrt{5} \text{ in}$$

$$\cos \beta = \frac{5\sqrt{5}}{18}$$

$$\beta = 51.60^\circ$$



Therefore,

$$A = \frac{1}{2} (AB)(AC) \sin(\alpha + \beta)$$

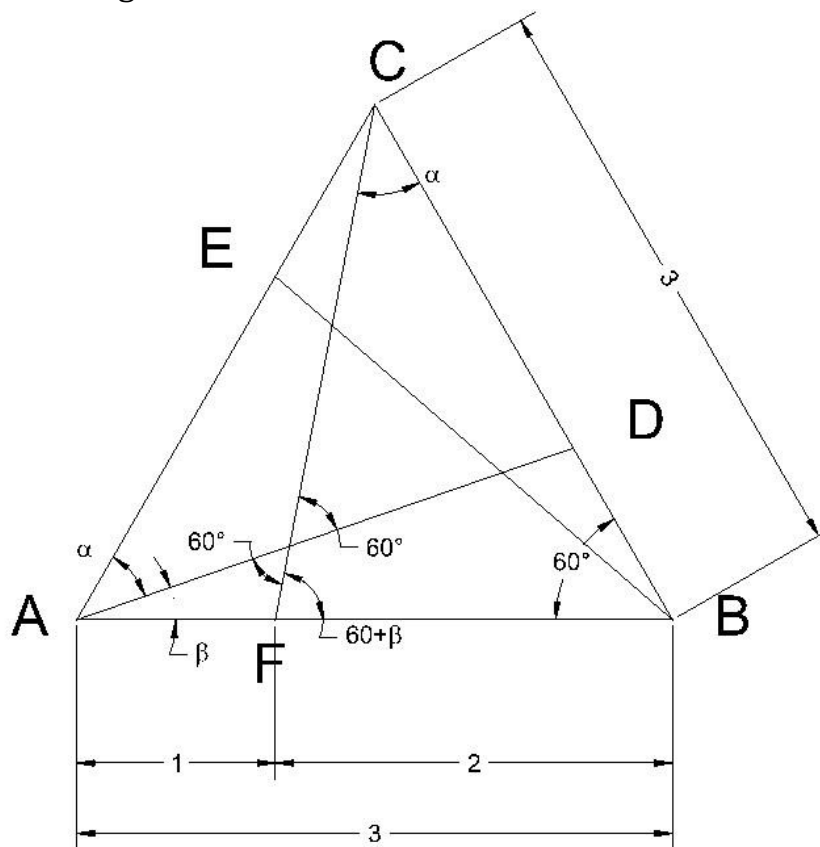
$$A = \frac{1}{2} (15)(18) \sin(41.81^\circ + 51.60^\circ)$$

$$A = 134.76 \text{ in}^2$$

6. If ΔABC is equilateral, $\frac{BD}{BC} = \frac{1}{3}$, $\frac{CE}{CA} = \frac{1}{3}$, and $\frac{AF}{AB} = \frac{1}{3}$. Find the ratio of the area of ΔABC to the area of the small triangle in the middle.

Solution:

Referring to the figure, let AF = 1, AB = 3



Consider ΔBCF , and solve for the length of CF by cosine law:

$$CF^2 = 2^2 + 3^2 - 2(2)(3) \cos 60^\circ$$

$$CF = \sqrt{7}$$

Again, consider $\triangle BCF$ and solve α using sine law:

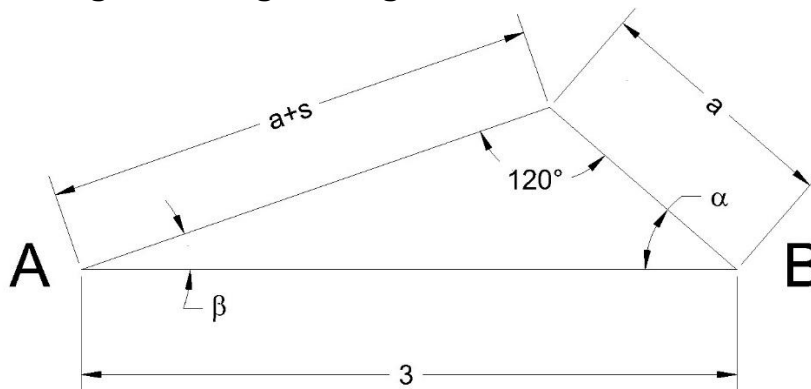
$$\frac{2}{\sin \alpha} = \frac{\sqrt{7}}{\sin 60^\circ}$$

$$\alpha = 40.89^\circ$$

The sum of α and β is equal to 60° , therefore:

$$\beta = 19.11^\circ$$

Now, considering this triangle, using sine law:



$$\frac{3}{\sin 120^\circ} = \frac{a}{\sin 19.11^\circ}$$

$$a = \frac{3\sqrt{7}}{7} \approx 1.133$$

$$\frac{3}{\sin 120^\circ} = \frac{1.133 + s}{\sin 40.89^\circ}$$

$$s = \frac{3\sqrt{7}}{7} \approx 1.133$$

Therefore, to solve for the ratio:

$$\frac{A_{\text{big}}}{A_{\text{small}}} = \left(\frac{3}{\frac{3\sqrt{7}}{7}} \right)^2$$

$$\frac{A_{\text{big}}}{A_{\text{small}}} = \frac{7}{1}$$

The ratio is **7:1**.

7. A side of a square is 16 inches. The midpoints of its sides are joined to form an inscribed square. Another square is drawn in such a way that its vertices would lie at the midpoints of the sides of the second square. This process is continued infinitely. Find the sum of the areas of these infinite squares.

Solution:

From the figure:

Since $AC = BC = 8$ inches, and $\angle C = 90^\circ$,

By Pythagorean Theorem:

$$8^2 + 8^2 = (AB)^2$$

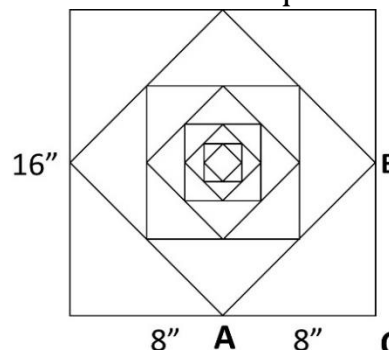
$$AB = 8\sqrt{2} \text{ inches}$$

$$\text{Area of first square} = 16^2 = 256 \text{ in}^2$$

$$\text{Area of second square} = (8\sqrt{2})^2 = 128 \text{ in}^2$$

Applying the same manner to the second square and third square,

$$\text{Area of third square} = 8^2 = 64 \text{ in}^2$$



It shows that the areas of the squares form an infinite geometric progression.

From the formula,

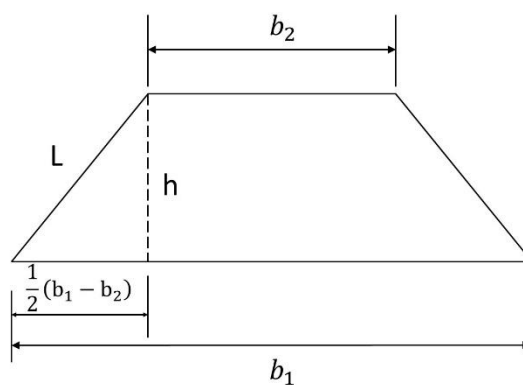
$$S_n = \frac{a_1}{1 - r}, r \neq 1$$

$$a_1 = 256, r = \frac{1}{2}$$

Hence,

$$S_n = \frac{256}{1 - \frac{1}{2}} = 512 \text{ in}^2$$

8. The area of an isosceles trapezoid is 246 m^2 . If the height and the length of one of its congruent sides measures 6 m and 10 m , respectively, find the lengths of the two bases.



Solution:

Referring to the figure:

$$A = \frac{1}{2} (b_1 + b_2)h$$

$$246 = \frac{1}{2} (b_1 + b_2)(6)$$

$$b_1 + b_2 = 82 \rightarrow \text{equation 1}$$

By Pythagorean Theorem:

$$\left(\frac{b_1 - b_2}{2}\right)^2 + h^2 = L^2$$

$$\left(\frac{b_1 - b_2}{2}\right)^2 + 6^2 = 10^2$$

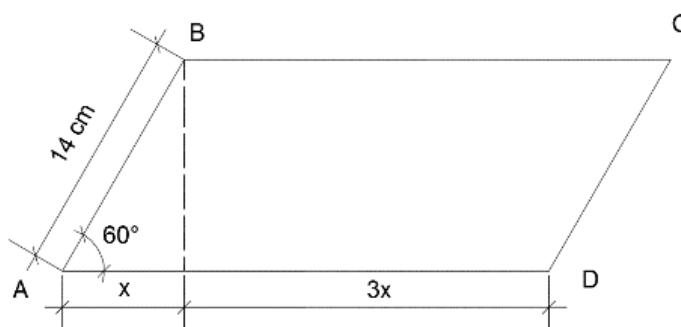
$$b_1 - b_2 = 16 \rightarrow \text{equation 2}$$

Solving simultaneously,

$$b_1 = 49 \text{ m}$$

$$b_2 = 33 \text{ m}$$

9. The altitude BE of parallelogram ABCD divides the side AD into segments in the ratio 1:3. Find the area of the parallelogram if the length of its shorter side AB is 14 cm , and one of its interior angle measures 60° .



Solution:

Referring to the figure:

$$\cos 60^\circ = \frac{x}{14}$$

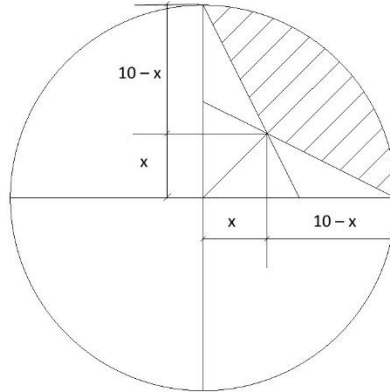
$$x = 7 \text{ cm}$$

$$\therefore AD = 4x = 28 \text{ cm}$$

$$A = bh = (14)(28) \sin 60^\circ = 339.48 \text{ cm}^2$$

10. Circle C has radius 10 cm. Each of points B and D is on the midpoint of the radius. Find the area of the shaded region. (See figure)

Solution:



By ratio and proportion:

$$\frac{x}{10 - x} = \frac{1}{2}$$

$$x = \frac{10}{3} \approx 3.33 \text{ cm}$$

To solve the area of the shaded region:

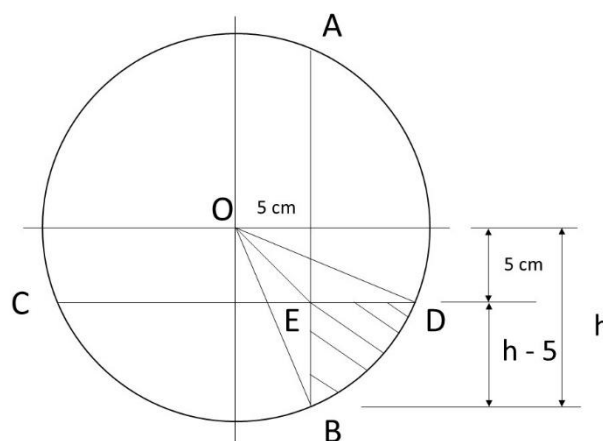
$$A_{\text{shaded}} = 2 \left(\frac{1}{2} (10)^2 \left(\frac{45\pi}{180} \right) - \frac{1}{2} (10) \left(\frac{10}{3} \sqrt{2} \right) \sin 45^\circ \right)$$

$$A_{\text{shaded}} = 45.21 \text{ cm}^2$$

11. Two perpendicular chords divide a circle with a radius 13 cm into four parts. If the perpendicular distances of both chords are 5 cm each from the center of the circle, find the area of the smallest part.

Solution:

Referring to the figure:



By Pythagorean Theorem:

$$5^2 + h^2 = 13^2$$

$$h = 12 \text{ cm}$$

Solving for OE:

$$5^2 + 5^2 = a^2$$

$$a = 5\sqrt{2}\text{cm}$$

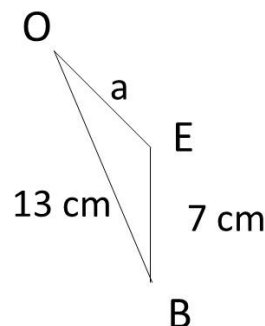
Isolate ΔOEB :

By cosine law:

$$7^2 = 13^2 + (5\sqrt{2})^2 - 2(13)(5\sqrt{2}) \cos \alpha$$

Solving for α :

$$\alpha = 22.38^\circ$$



Therefore,

$$A_{\text{shaded}} = A_{\text{sector}} - 2A_{\Delta OEB}$$

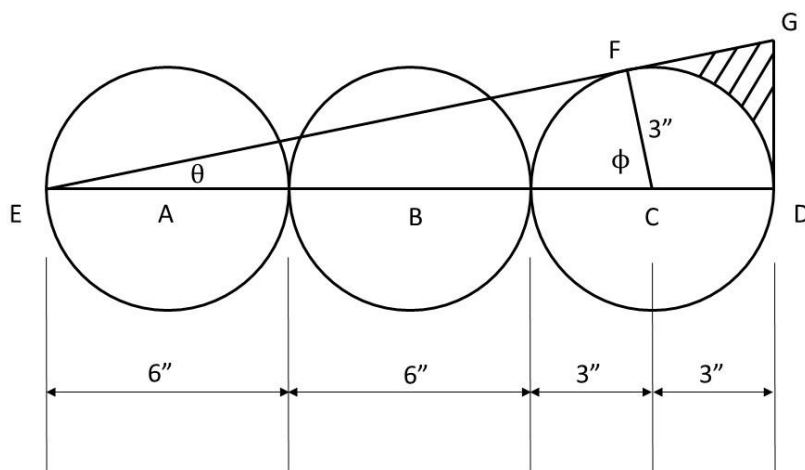
$$A_{\text{shaded}} = \frac{1}{2} (13)^2 \left[2(22.38^\circ) \left(\frac{\pi}{180^\circ} \right) \right] - 2 \left[\frac{1}{2} (13)(5\sqrt{2}) \sin 22.38^\circ \right]$$

$$A_{\text{shaded}} = \mathbf{31.01 \text{ cm}^2}$$

12. Three circles with centers A, B, and C are externally tangent to each other as shown in the figure. Lines EG and DG are tangent to circle C at points F and D and intersect at point G. If each circle has a diameter of 6 inches, find the area enclosed by lines FG and GD, and arc FD. (See figure)

Solution:

Referring to the figure:



$$\sin \theta = \frac{3}{15}$$

$$\theta = 11.54^\circ$$

But,

$$\theta + \phi = 90^\circ$$

$$\phi = 78.46^\circ; 180 - \phi = 101.54^\circ$$

$$\tan \theta = \frac{DG}{18}$$

$$DG = 18 \tan 11.54^\circ = 3.674 \text{ in}$$

By Pythagorean Theorem,

$$(EF)^2 + 3^2 = 15^2$$

$$EF = 6\sqrt{6}\text{in}$$

Therefore,

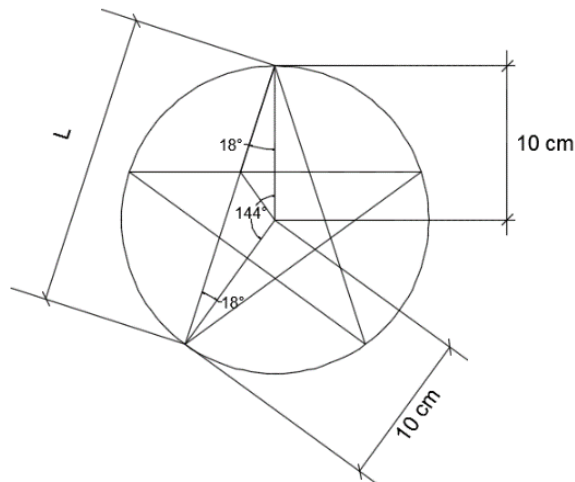
$$A_{\text{shaded}} = A_{\Delta ADG} - A_{\Delta AFC} - A_{\text{sector FD}}$$

$$A_{\text{shaded}} = \frac{1}{2} (18)(3.674) - \frac{1}{2} (3)(6\sqrt{6}) - \frac{1}{2} (3)^2 \left(101.54^\circ \cdot \frac{\pi}{180^\circ} \right)$$

$$A_{\text{shaded}} = \mathbf{3.048 \text{ in}^2}$$

13. In a circle with diameter of 20 cm, a regular five-pointed star touching its circumference is inscribed. Find the area of the star.

Solution:



Referring to the isolated figure:

By sine law:

$$\frac{10}{\sin 126^\circ} = \frac{x}{\sin 18^\circ}$$

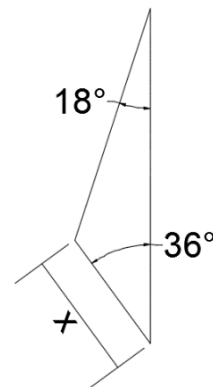
$$x = 3.82 \text{ cm}$$

Therefore,

$$A_{\text{pentagram}} = 10 \cdot A_{\Delta AOB}$$

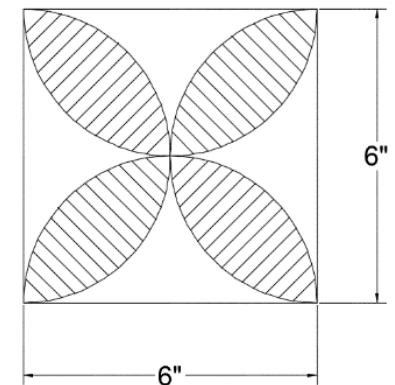
$$A_{\text{pentagram}} = 10 \left[\frac{1}{2} (3.82)(10) \sin 36^\circ \right]$$

$$A_{\text{pentagram}} = 112.26 \text{ cm}^2$$



14. If the area of the square shown in the figure is 36 square inches, find the area of the inscribed four-leaf figure. (See figure)

Solution:



From the figure:

$$-A_{\text{shaded}} = A_{\text{square}} - 4A_{\text{semicircle}}$$

$$A_{\text{shaded}} = 4A_{\text{semicircle}} - A_{\text{square}}$$

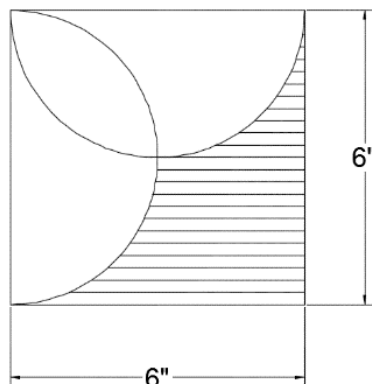
$$A_{\text{shaded}} = 4 \left[\frac{\pi}{8} (6)^2 \right] - 6^2$$

$$A_{\text{shaded}} = 18\pi - 36$$

$$A_{\text{shaded}} = 20.55 \text{ in}^2$$

15. Two semicircles are inscribed in a square with side 8 cm as shown. Find the area of the shaded region. (See figure)

Solution:



Referring to the figure:

$$A_{\text{shaded}} = A_{\text{square}} - (A_1 + A_2 + A_3)$$

$$A_{\text{shaded}} = 8^2 - \left[\frac{\pi}{4}(4)^2 + \frac{\pi}{4}(4)^2 + 4^2 \right]$$

$$A_{\text{shaded}} = 22.87 \text{ cm}^2$$

16. The corresponding edges of two regular tetrahedrons are 1 cm and 3 cm. If the sum of the weights of the two tetrahedrons is 100 grams and both solids are made up of the same material, find the weight of the bigger solid.

Solution:

Remember that:

$$W = \gamma V$$

Since the solids are made up of the same material, γ is constant. Therefore,

$$V = \frac{W}{\gamma}$$

By ratio and proportion:

$$\frac{V_{\text{small}}}{V_{\text{big}}} = \left(\frac{S_{\text{small}}}{S_{\text{big}}} \right)^3$$

$$\frac{V_{\text{small}}}{V_{\text{big}}} = \left(\frac{1}{3} \right)^3$$

$$\frac{\frac{W_{\text{small}}}{\gamma}}{\frac{W_{\text{big}}}{\gamma}} = \frac{1}{27}$$

$$\frac{W_{\text{small}}}{W_{\text{big}}} = \frac{1}{27}$$

$$W_{\text{big}} = 27W_{\text{small}}$$

$$W_{\text{big}} - 27W_{\text{small}} = 0 \rightarrow \text{equation 1}$$

From the other given,

$$W_{\text{big}} + W_{\text{small}} = 100 \rightarrow \text{equation 2}$$

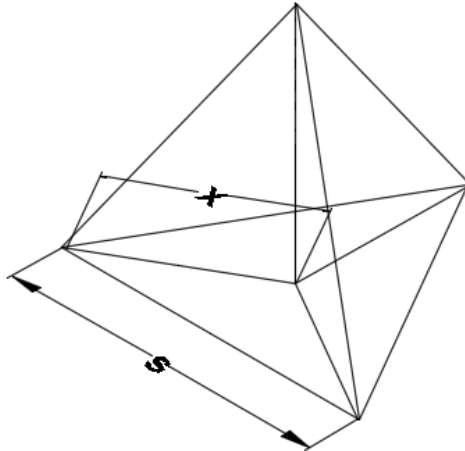
Solving simultaneously,

$$\begin{cases} W_{\text{big}} = 96.43 \text{ g} \\ W_{\text{small}} = 3.57 \text{ g} \end{cases}$$

17. Find the volume of a regular tetrahedron whose altitude measures 10 cm.

Solution:

Referring to the figure:



By Pythagorean Theorem:

$$x^2 + h^2 = s^2$$

$$x^2 + 10^2 = s^2 \rightarrow \text{equation 1}$$

By cosine law:

$$s^2 = x^2 + x^2 - 2(x)(x) \cos 120^\circ$$

$$s^2 = 3x^2$$

$$\frac{s^2}{3} = x^2 \rightarrow \text{equation 2}$$

Substitute equation 2 to equation 1:

$$\frac{s^2}{3} + 10^2 = s^2$$

$$s = 5\sqrt{6} \text{ cm}$$

Therefore, the volume of the tetrahedron is:

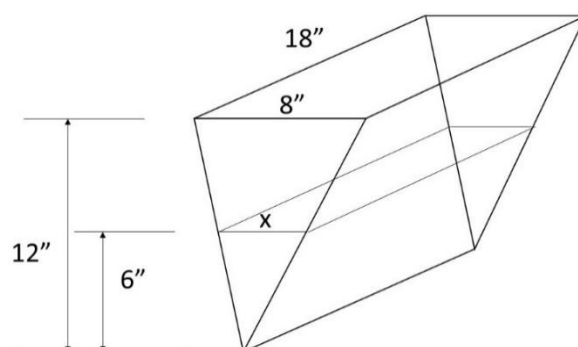
$$V = \frac{1}{3} A_{\text{BASE}} H$$

$$V = \frac{1}{3} \left(\frac{(5\sqrt{6})^2 \sqrt{3}}{4} \right) (10)$$

$$V = 216.51 \text{ cm}^3$$

18. A container in the form of a triangular prism as shown in the figure below contains water at a depth of 6 inches. Find the volume of water. (See figure)

Solution:



By similar triangles,

$$\frac{x}{6} = \frac{8}{12}; x = 4"$$

Therefore,

$$V_{H_2O} = V_{\text{wet prism}}$$

$$V_{H_2O} = \left[\frac{1}{2} (4)(6) \right] (18)$$

$$V_{H_2O} = \mathbf{216 \text{ in}^3}$$

19. A flower vase, in the form of a hexagonal prism, is to be filled with 512 cubic inches of water. Find the height of the water if the wet portion of the flower vase and its volume are numerically equal.

Solution:

Take note that the top surface is not wet.

$$\text{Wetted perimeter} = \text{Lateral Surface Area} = 512 \text{ in}^2$$

$$\text{Volume} = 512 \text{ in}^3$$

$$V = A_{\text{base}} h$$

$$V = 6 \cdot \frac{s^2 \sqrt{3}}{4} h$$

$$512 = 6 \cdot \frac{s^2 \sqrt{3}}{4} h$$

$$h = \frac{1024}{3\sqrt{3}s^2} \rightarrow \text{equation 1}$$

$$\text{Wet SA} = A_{\text{base}} + 6sh$$

$$512 = 6 \cdot \frac{s^2 \sqrt{3}}{4} + 6sh \rightarrow \text{equation 2}$$

Substitute equation 1 to equation 2:

$$512 = \frac{3s^2 \sqrt{3}}{2} + 6s \left(\frac{1024}{3\sqrt{3}s^2} \right)$$

Simplifying,

$$\frac{3s^3 \sqrt{3}}{2} - 512s + \frac{2048}{\sqrt{3}} = 0$$

Go to MODE 5-4 and get the roots:

$$\begin{cases} s_1 = -15.08 \text{ (not valid)} \\ s_2 = 12.70 \text{ (ok)} \\ s_3 = 2.38 \text{ (ok)} \end{cases}$$

Substituting the value on equation 1:

$$\left\{ \begin{array}{l} h = \frac{1024}{3\sqrt{3}(12.70)^2} \quad \left| \quad h = \frac{1024}{3\sqrt{3}(2.38)^2} \right. \\ h = 1.22 \text{ in} \quad \quad \quad \left| \quad h = 34.86 \text{ in} \right. \end{array} \right.$$

Since the problem stated it is a flower vase, it is impractical to answer $h = 1.22$ inches, therefore, the answer is **$h = 34.86$ inches.**

20. In a regular hexagonal prism, the ratio of an edge of the base to the height of the prism is 1:5. Find the volume of the prism if the lateral area is 270 cm².

Solution:

$$\frac{s}{h} = \frac{1}{5} \rightarrow h = 5s$$

$$A = 6sh = 6s(5s)$$

$$A = 30s^2$$

$$270 = 30s^2$$

$$s^2 = 9$$

$$s = 3 \text{ cm}$$

Therefore,

$$V = BH$$

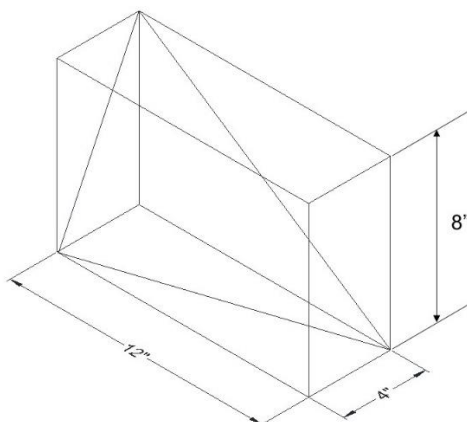
$$V = 6 \cdot \frac{s^2\sqrt{3}}{4} (5s)$$

$$V = 6 \cdot \frac{3^2\sqrt{3}}{4} (5)(3)$$

$$V = 350.74 \text{ cm}^3$$

21. Find the area of the triangular section of the rectangular solid shown in the figure. (See figure)

Solution:



Using Heron's Formula:

Using multi-lineing equation in your calculator:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Go to MODE 1 and type:

$$X = \frac{A + B + C}{2} : \sqrt{X(X-A)(X-B)(X-C)}$$

Then press CALC:

$$A = \sqrt{4^2 + 8^2}$$

$$B = \sqrt{4^2 + 12^2}$$

$$C = \sqrt{8^2 + 12^2}$$

Then press =

$$X = 18.00779383$$

$$\therefore 56 \text{ in}^2$$

22. Find the edge of a cube if its surface area is numerically equal to its volume.

Solution:

$$\text{TSA} = 6s^2$$

$$V = s^3$$

Since the surface area and volume are numerically equal:

$$6s^2 = s^3$$

$$\therefore s = 6 \text{ units}$$

23. The base of a rectangular solid is 4 ft. long and 3 ft. wide. Find the volume of the solid if its diagonal is $\sqrt{41}$ ft.

Solution:

$$d^2 = 3^2 + 4^2 + h^2$$

$$(\sqrt{41})^2 = 3^2 + 4^2 + h^2$$

$$h = 4'$$

Therefore,

$$V = lwh$$

$$V = (4)(3)(4)$$

$$V = 48 \text{ ft}^3$$

24. The size of a brick is 25 cm x 12 cm x 55 cm and its density is $1.8 \times 10^3 \text{ kg/m}^3$. How many trips must a 3-ton dumper carry out to transport 15,000 bricks to a construction site? (Hint: 1 metric ton is equivalent to 1,000 kg.)

Solution:

$$W_{\text{brick}} = \left(\frac{1.8 \times 10^3 \text{ kg}}{\text{m}^3} \right) (0.25\text{m})(0.12\text{m})(0.55\text{m})$$

$$W_{\text{brick}} = 29.7 \text{ kg}$$

$$\therefore 15000 \text{ bricks} = 445500 \text{ kg}$$

Divide it by 3 tons = 3000 kg to get how many trips are required.

Therefore,

$$\text{number of trips} = \frac{445500 \text{ kg}}{3000 \text{ kg/trip}} = 148.5 \text{ say } \mathbf{149 \text{ trips}}$$

25. A cylindrical tank of height equal to twice the diameter of its base can hold 10 liters ($1 \text{ L} = 1,000 \text{ cm}^3$) of water. Another cylindrical container with the same capacity has its height equal to three times the diameter of its base. Find the ratio of the amount of aluminum required for making the two containers, including covers.

Solution:

For container 1:

$$2D_1 = h_1$$

Since it can hold 10 L = 10000 cm^3 :

$$V_1 = \pi R_1^2 h_1$$

$$V_1 = \frac{\pi}{4} D_1^2 h_1$$

$$10000 = \frac{\pi}{4} (D_1)^2 (2D_1)$$

$$\therefore D_1 = 18.53 \text{ cm}$$

For container 2:

$$3D_2 = h_2$$

Since it can also hold 10 L = 10000 cm³:

$$V_2 = \pi R_2^2 h_2$$

$$V_2 = \frac{\pi}{4} D_2^2 h_2$$

$$10000 = \frac{\pi}{4} (D_2)^2 (3D_2)$$

$$\therefore D_2 = 16.19 \text{ cm}$$

For the total surface area of container 1:

$$TSA_1 = 2 \cdot \frac{\pi}{4} D_1^2 + \pi D_1 h_1$$

$$TSA_1 = \frac{\pi}{2} D_1^2 + \pi D_1 h_1$$

$$TSA_1 = \frac{\pi}{2} (18.53)^2 + \pi (18.53) (2 \cdot 18.53)$$

$$TSA_1 = 2697.80 \text{ cm}^2$$

For the total surface area of container 2:

$$TSA_2 = 2 \cdot \frac{\pi}{4} D_2^2 + \pi D_2 h_2$$

$$TSA_2 = \frac{\pi}{2} D_2^2 + \pi D_2 h_2$$

$$TSA_2 = \frac{\pi}{2} (16.19)^2 + \pi (16.19) (3 \cdot 16.19)$$

$$TSA_2 = 2882.33 \text{ cm}^2$$

Therefore,

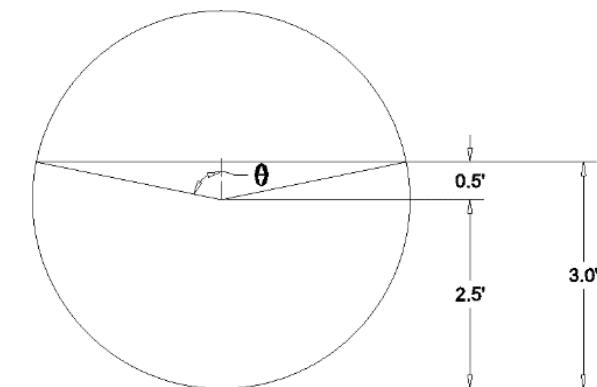
$$\frac{TSA_1}{TSA_2} = \frac{2697.80}{2882.33}$$

$$\frac{TSA_1}{TSA_2} = \mathbf{0.935}$$

26. A closed cylindrical tank measures 12 ft long and 5 ft in diameter. It has to contain water with a depth of 3 feet when the tank is lying in horizontal position. Find the height of water when the tank is in its vertical position.

Solution:

Referring to the figure:



$$\cos \theta = \frac{0.5}{2.5}$$

$$\theta = 78.46^\circ$$

Therefore,

$$A_o = A_{\text{circle}} - A_{\text{segment}}$$

$$A_o = A_{\text{circle}} - (A_{\text{sector}} - A_{\Delta})$$

$$A_o = \pi(2.5)^2 - \left[\frac{1}{2} (2.5)^2 \left(\frac{2 \times 78.46\pi}{180} \right) - \frac{1}{2} (2.5) \sin(2 \times 78.46^\circ) \right]$$

$$A_o = 12.30 \text{ ft}^2$$

Therefore,

$$V = A_o h$$

$$V = (12.30)(12)$$

$$V = 147.61 \text{ ft}^3$$

When held in vertical position:

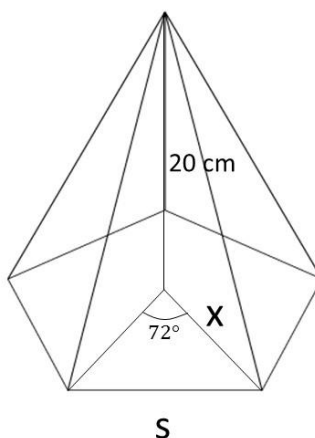
$$V = \pi r^2 h$$

$$147.61 = \pi(2.5)^2 h$$

$$\mathbf{h = 7.52 \text{ ft}}$$

27. A regular pentagonal pyramid has an altitude of 20 cm and a slant height which measures 25 cm. Find the volume of the pyramid.

Solution:



Referring to the figure:

By Pythagorean Theorem:

$$x^2 + 20^2 = 25^2$$

$$x = 15 \text{ cm}$$

From the base of the pyramid:

Using cosine law:

$$s^2 = x^2 + x^2 - 2x^2 \cos 72^\circ$$

$$s^2 = 15^2 + 15^2 - 2(15)^2 \cos 72^\circ$$

$$s = 17.63 \text{ cm}$$

Therefore,

$$V = \frac{1}{3} BH$$

$$V = \frac{1}{3} \left[\frac{ns^2}{4} \cot \left(\frac{180^\circ}{n} \right) \right] (H)$$

$$V = \frac{1}{3} \left[\frac{5(17.63)^2}{4} \cot \left(\frac{180^\circ}{5} \right) \right] (20)$$

$$\mathbf{V = 3566.46 \text{ cm}^3}$$

28. Find the volume of the pyramid bounded by the plane defined by $2x + 3y + z = 6$ and the coordinate planes.

Solution:

Rewrite the equation into the form:

$$2x + 3y + z = 6 \div 6$$

$$\frac{x}{3} + \frac{y}{2} + \frac{z}{6} = 1$$

Therefore,

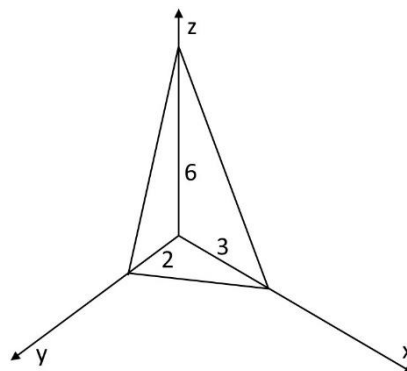
$$\begin{cases} x - \text{intercept} = 3 \\ y - \text{intercept} = 2 \\ z - \text{intercept} = 6 \end{cases}$$

Referring to the figure,

$$V = \frac{1}{3}BH$$

$$V = \frac{1}{3} \left[\frac{1}{2} (3)(2) \right] (6)$$

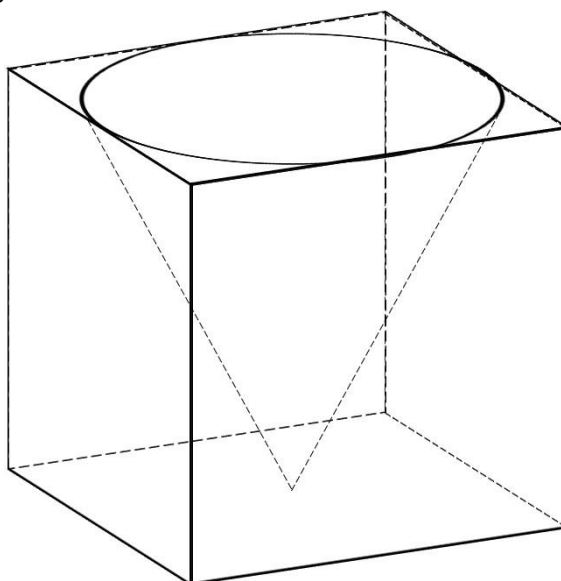
$$V = 6 \text{ cubic units}$$



29. A right circular cone is inscribed in a cube having a diagonal which measures $10\sqrt{3}$ cm. Find the volume of the cone.

Solution:

Referring to the figure:



$$d^2 = s^2 + s^2 + s^2$$

$$d^2 = 3s^2$$

$$(10\sqrt{3})^2 = 3s^2$$

$$s = 10 \text{ cm}$$

Therefore,

$$V_{\text{cone}} = \frac{1}{3}BH$$

$$V_{\text{cone}} = \frac{1}{3}\pi R^2 H$$

$$V_{\text{cone}} = \frac{1}{3}[\pi(5)^2](10)$$

$$V_{\text{cone}} = \frac{250}{3}\pi \text{ cm}^3$$

30. Two similar cones have volumes $\frac{81\pi}{2} \text{ in}^3$ and $12\pi \text{ in}^3$, and the slant height of the bigger cone is 7.5 in. Find the integer solution to the slant height of the smaller cone.

Solution:

By ratio and proportion:

$$\frac{V_{\text{big}}}{V_{\text{small}}} = \left(\frac{l_{\text{big}}}{l_{\text{small}}} \right)^3$$

$$\frac{\frac{81}{2}\pi}{12\pi} = \left(\frac{7.5}{l_{\text{small}}} \right)^3$$

$$l_{\text{small}} = 5 \text{ in}$$

31. A steel cylinder has a 4-inch diameter base and a height of 6 inches. A hole in the form of a right circular cone with its base coincident with the base of the cylinder and axis coincident with the axis of the cylinder is filled with lead. The diameter of the base of the cone is 2 inches and its height is 1 inch. If the steel weighs 490 lb/ft^3 and the lead weighs 710 lb/ft^3 , find the total weight of the composite solid.

Solution:

Referring to the figure:

$$\gamma_{\text{steel}} = 490 \frac{\text{lb}}{\text{ft}^3} \cdot \left(1 \frac{\text{ft}}{12 \text{ in}} \right)^3 = \frac{245 \text{ lb}}{864 \text{ in}^3}$$

$$\gamma_{\text{lead}} = 710 \frac{\text{lb}}{\text{ft}^3} \cdot \left(1 \frac{\text{ft}}{12 \text{ in}} \right)^3 = \frac{355 \text{ lb}}{864 \text{ in}^3}$$

Therefore,

$$W_{\text{total}} = W_{\text{steel}} + W_{\text{lead}}$$

$$W_{\text{total}} = V_{\text{steel}}\gamma_{\text{steel}} + V_{\text{lead}}\gamma_{\text{lead}}$$

$$W_{\text{total}} = [(V_{\text{cylinder}} - V_{\text{cone}})\gamma_{\text{steel}}] + V_{\text{cone}}\gamma_{\text{lead}}$$

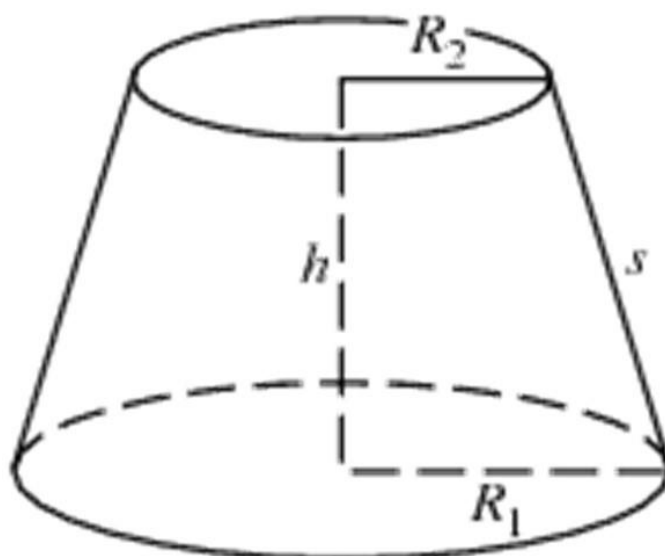
$$W_{\text{total}} = \left[\pi(2)^2(6) - \frac{1}{3}\pi(1)^2(1) \right] \left(\frac{245}{864} \right) + \frac{1}{3}\pi(1)^2(1) \left(\frac{355}{864} \right)$$

$$W_{\text{total}} = 21.51 \text{ lbs}$$

32. The volume of a frustum of a right circular cone is $52\pi \text{ ft}^3$. Its altitude is 3 ft and the measure of its lower radius is three times the measure of its upper radius. Find the lateral area of the frustum.

Solution:

Referring to the figure:



$$\begin{aligned}
 V &= 52\pi \text{ ft}^3 \\
 R_1 &= 3R_2 \\
 V &= \frac{h}{3} \left(A_{\text{lower}} + A_{\text{upper}} + \sqrt{A_{\text{lower}} \times A_{\text{upper}}} \right) \\
 512\pi &= \frac{3}{3} \left[\pi(R_1)^2 + \pi(R_2)^2 + \sqrt{\pi(R_1)^2 \cdot \pi(R_2)^2} \right] \\
 512\pi &= \left[\pi(3R_2)^2 + \pi(R_2)^2 + \sqrt{\pi(3R_2)^2 \cdot \pi(R_2)^2} \right] \\
 512\pi &= 13\pi R_2^2 \\
 R_2 &= \frac{16\sqrt{26}}{13} = 6.28 \text{ ft} \\
 R_1 &= 3R_2 = \frac{48\sqrt{26}}{13} = 18.83 \text{ ft}
 \end{aligned}$$

By Pythagorean Theorem:

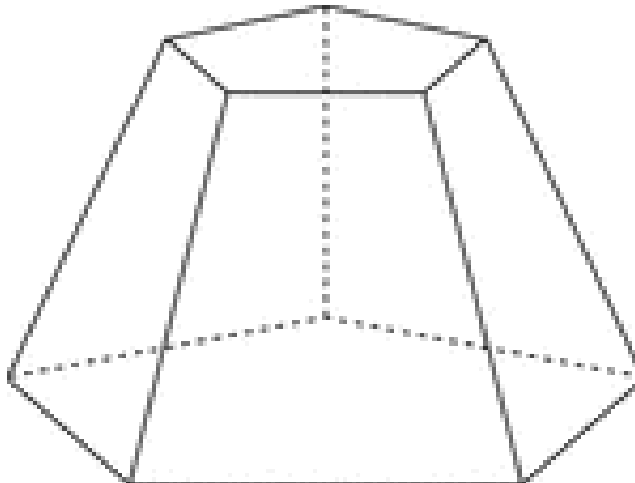
$$\begin{aligned}
 (R_1 - R_2)^2 + 3^2 &= s^2 \\
 s^2 &= (18.83 - 6.28)^2 + 3^2 \\
 s &= 12.90 \text{ ft}
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \text{LSA} &= \frac{1}{2} (P_1 + P_2)s \\
 \text{LSA} &= \frac{1}{2} [2\pi(6.28) + 2\pi(18.83)](12.90) \\
 \text{LSA} &= \mathbf{1017.72 \text{ ft}^2}
 \end{aligned}$$

33. The base edges of a frustum of a regular pentagonal pyramid are 4 in and 8 in, and its altitude is 10 in. Find the volume of the frustum.

Solution:



Referring to the figure:

$$\begin{aligned}
 A_{\text{lower}} &= \frac{ns_2^2}{4} \cot\left(\frac{180^\circ}{n}\right) = \frac{5(8)^2}{4} \cot\left(\frac{180^\circ}{5}\right) = 110.11 \text{ in}^2 \\
 A_{\text{upper}} &= \frac{ns_1^2}{4} \cot\left(\frac{180^\circ}{n}\right) = \frac{5(4)^2}{4} \cot\left(\frac{180^\circ}{5}\right) = 27.53 \text{ in}^2
 \end{aligned}$$

$$\begin{aligned}
 V &= \frac{H}{3} \left(A_{\text{lower}} + A_{\text{upper}} + \sqrt{A_{\text{lower}} \times A_{\text{upper}}} \right) \\
 V &= \frac{10}{3} [27.53 + 110.11 + \sqrt{27.53 \times 110.11}] \\
 V &= \mathbf{642.31 \text{ in}^2}
 \end{aligned}$$

34. The volume of a truncated right triangular prism is 72 ft^3 . The lengths of the two perpendicular edges of the base are 3 ft and 4 ft, and two of the lateral edges measure 14 ft and 12 ft. Find the measure of the third lateral edge.

Solution:

Referring to the figure:

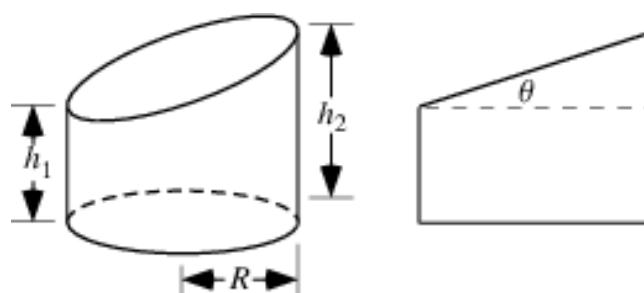
$$V = BH$$

$$72 = \left[\frac{1}{2} (3)(4) \right] \left[\frac{12 + 14 + h}{3} \right]$$

$$\mathbf{h = 10 \text{ ft}}$$

35. In a truncated right circular cylinder, the top plane makes an angle of 60° with the horizontal, and the shortest and longest elements are 4 and 10 units, respectively. Find the volume of the solid.

Solution:



Referring to the figure:

$$h_1 = 4; h_2 = 10; \theta = 60^\circ$$

$$\tan \theta = \frac{h_2 - h_1}{D}$$

$$\tan 60^\circ = \frac{10 - 4}{D}$$

$$\tan 60^\circ = \frac{6}{D}$$

$$D = 2\sqrt{3} \text{ units}$$

Therefore, we have:

$$V = BH$$

$$V = \frac{\pi D^2}{4} \left(\frac{h_1 + h_2}{2} \right)$$

$$V = \frac{\pi (2\sqrt{3})^2}{4} \left(\frac{4 + 10}{2} \right)$$

$$\mathbf{V = 21\pi \text{ cubic units}}$$

36. Suppose two cylindrical tubes with the same diameter intersect at right angles. If the diameter of each tube is 6 cm, find the volume of the common part between the tubes.

Solution:

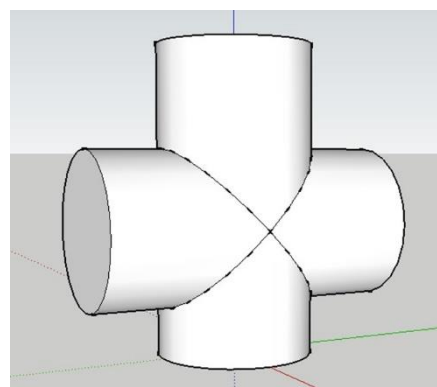
Referring to the figure:

By Prismatoid Formula:

$$V = \frac{L}{6} [A_1 + 4A_m + A_2]$$

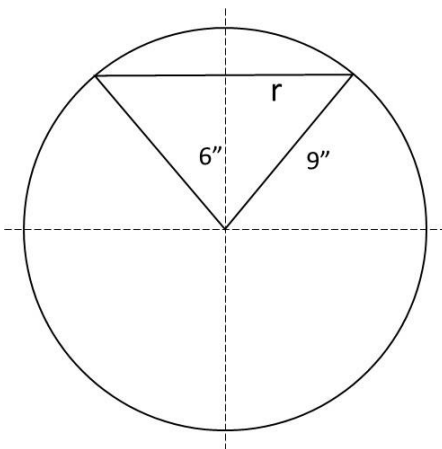
$$V = \frac{6}{6} [0 + 4(6)^2 + 0]$$

$$\mathbf{V = 144 \text{ cm}^3}$$



37. The plane of a small circle in a sphere of diameter 18 inches is 6 inches from the center of the sphere. Find the area of the small circle.

Solution:



Referring to the figure:

By Pythagorean Theorem:

$$r^2 + 6^2 = 9^2$$

$$r^2 = 45 \text{ in}^2$$

Therefore,

$$A = \pi r^2$$

$$A = \pi(45)$$

$$A = 45\pi \text{ in}^2$$

38. A plane is tangent to a sphere at point A. A point B on this plane is 10 cm from the surface of the sphere and is 16 cm from point A. Find the surface area of the sphere.

Solution:

Referring to the figure:

By Pythagorean Theorem:

$$r^2 + 16^2 = (10 + r)^2$$

Solving for r, we have:

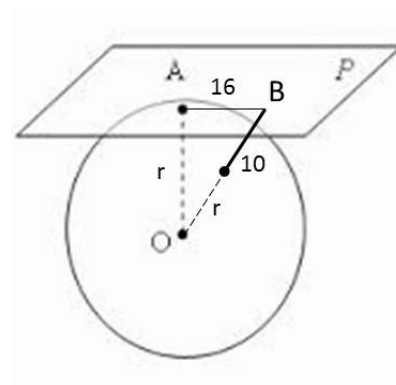
$$r = 7.8 \text{ cm}$$

Therefore, from the formula of surface area of the sphere, we have:

$$S = 4\pi r^2$$

$$S = 4\pi(7.8)^2$$

$$S = 764.54 \text{ cm}^2$$



39. The base of a spherical pyramid is a bi-rectangular triangle whose third angle is 23° . If the diameter of the sphere is 90 cm, find the volume of the spherical pyramid.

Solution:

Note: A bi-rectangular triangle has two of its angles equal to 90° each.

$$V = \frac{\pi r^3 E}{540}$$

$$E = (A + B + C) - 180^\circ$$

$$E = (90^\circ + 90^\circ + 23^\circ) - 180^\circ$$

$$E = 23^\circ$$

$$V = \frac{\pi(45)^2(23)}{540}$$

$$V = 12193 \text{ cm}^3$$

SPACE ANALYTIC GEOMETRY

- Find the distance between the points A (1,9, -6) and B (1, 8, -7).
 A. 1.313
 b. 1.414
 C. 1.732
 D. 1.778
- Find the angle between the lines AB and CD. A (-3, 2, 4), B (2, 5, -2) and C (1, -2, 2), D (4, 2, 3).
 A. 62.44°
 B. 58.55°
 C. 65.43°
 D. 60.51°
- Find the Direction Cosine of the line that is perpendicular to each of the two lines whose directions are $\langle 3 \ 0 \ 1 \rangle$ and $\langle 2 \ -1 \ 2 \rangle$.
 A. $\langle 0.196 \ 0.784 \ -0.858 \rangle$
 B. $\langle 0.169 \ -0.784 \ 0.585 \rangle$
 C. $\langle 0.196 \ -0.784 \ -0.588 \rangle$
 D. $\langle 0.169 \ -0.784 \ -0.588 \rangle$
- Determine the value of k and q so that ABC are collinear A(1, 0, -2), B(3, -1, 1) and C(k, -3, q) are collinear.
 A. -7, 0
 B. 7, 7
 C. 0, 5
 D. 5, 7
- Find the coordinates of the foot of the perpendicular from A (1, 1, 1) on the line joining B (1, 4, 6) and C (5, 4, 4).
 A. (5, 12, 13)
 B. (3, 4, 5)
 C. (3, -4, 5)
 D. (4, 3, 5)
- Find the coordinates of the foot of the perpendicular drawn from the point (3,4,5) to the x-axis.
 A. (3, 0, 0)
 B. (3, 0, 5)
 C. (3, -4, 0)
 D. (3, 4, 0)
- Find the coordinates of the point where the segment joining the points (2, -2, 1) and (5, 1, -2) crosses the xy-plane.
 A. (3, 1, 0)
 B. (-3, 1, 0)
 C. (3, -1, 0)
 D. (-3, -1, 0)
- Find the points that trisect the segment joining the points (3, -1, 5) and (0, 5, -4).
 A. (2, -1, 3), (1, 3, 2)
 B. (2, 1, 2), (1, 3, -1)
 C. (-2, 1, 2), (-3, 1, 1)
 D. (1, 3, 1), (2, 1, 2)
- Find the equation of the plane through P (4, 3, 6) and perpendicular to the line joining P (4, 3, 6) to the point Q (2, 3, 1).
 A. $2x + 5z - 38 = 0$
 B. $5x + 2z - 12 = 0$
 C. $5x - 2z + 45 = 0$
 D. $3x + 5z - 20 = 0$
- Find the equation of the plane through the point P (1, 2, -1) and parallel to the plane $2x - 3y + 4z + 6 = 0$.
 A. $2x - 3y + 4z + 8 = 0$
 B. $3x + y - 8z + 6 = 0$
 C. $7x - 6y + 4z = 0$
 D. $5x + 5y + 2z + 15 = 0$

11. Find the equation of the plane that contains the three points P (1, -2, 4), Q (4, 1, 7), R (-1, 5, 1).
- A. $2x + 5y + 6z + 12 = 0$ C. $10x - 2y - 9z - 24 = 0$
 B. $10x - y - 9z + 24 = 0$ D. $2x + 5y + 6z - 12 = 0$
12. Find the perpendicular distance between the point (-2, 8, -3) and plane $9x - y - 4z + 1 = 0$.
- A. 1.313 C. 1.732
 B. 1.414 D. 1.778
13. Find the perpendicular distance between the two planes $x - 2y + 2z = 6$, $3x - 6y + 6z = 2$.
- A. 2.828 C. 1.732
 B. 1.414 D. 1.778
14. Find the equation of the line through the following points (1, 3, -2), (2, 2, 0), in symmetric form.
- A. $\frac{x+1}{1} = \frac{y-3}{1} = \frac{z+2}{2}$ C. $\frac{x-1}{1} = \frac{y-3}{-1} = \frac{z+2}{2}$
 B. $\frac{x-1}{1} = \frac{y-3}{3} = \frac{z+2}{2}$ D. $\frac{x-1}{1} = \frac{y-3}{1} = \frac{z-2}{-2}$
15. Find the equation of the line through the following points (1, 3, -2), (2, 2, 0), in parametric form.
- A. $x = t + 1, y = -t + 3, z = 2t - 2$ C. $x = t - 1, y = t + 3, z = 2t + 2$
 B. $x = -t + 1, y = 2t - 3, z = t + 2$ D. $x = t + 1, y = t - 3, z = 2t - 1$
16. Find the point on the line $x = y = z$, that is equidistant from the points (3, 0, 5) and (1, -1, 4).
- A. (0, 0, 0) C. (1, 1, 1)
 B. (-2, -2, -2) D. (2, 2, 2)
17. Find the area of the triangle whose sides have the equations
- $$\frac{x}{3} = \frac{y}{4} = \frac{z}{5}; \frac{x}{2} = \frac{y}{1} = \frac{z}{-2}; \frac{x}{2} = \frac{y+5}{3} = \frac{z+16}{7}$$
- A. 251.95 C. 242.44
 B. 305.51 D. 320.71
18. Find the equation of a line formed if Vector A = $\langle 1 \ 2 \ 3 \rangle$ is made to pass at point (2,9,8).
- A. $\frac{x+2}{1} = \frac{y+9}{2} = \frac{z+8}{3}$ C. $\frac{x-1}{2} = \frac{y-2}{9} = \frac{z-3}{8}$
 B. $\frac{x-2}{1} = \frac{y-9}{2} = \frac{z-8}{3}$ D. $\frac{x+1}{2} = \frac{y+2}{9} = \frac{z+3}{8}$
19. Find the center and radius of the sphere whose equation is $x^2 + y^2 + z^2 = 6x + 8z$.
- A. (-3, 0, -4); r = 5 C. (-3, 0, -4); r = 25
 B. (3, 0, 4); r = 5 D. (3, 0, 4); r = 25
20. Find the equation of the sphere tangent to the plane $6x + 6y + 7z = 22$, and has its center at origin and point of tangency.
- A. $x^2 + y^2 + z^2 = 1$ C. $x^2 + y^2 + z^2 = 4$
 B. $x^2 + y^2 + z^2 = 9$ D. $x^2 + y^2 + z^2 = 16$

SPACE ANALYTIC GEOMETRY SOLUTIONS

1. Find the distance between the points A (1, 9, -6) and B (1, 8, -7).

Solution:

Distance from A to B:

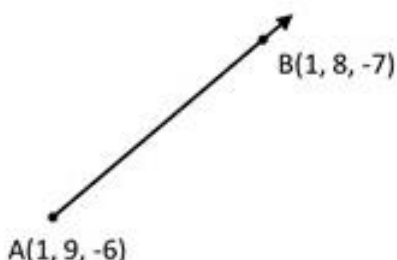
Using the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$d = \sqrt{(1 - 1)^2 + (8 - 9)^2 + (-7 - (-6))^2}$$

$$d = \sqrt{2} \approx 1.414$$

Or using the fact that the length of AB is the absolute value of vector \overline{AB} , therefore $d = |\overline{AB}|$:



$$\overline{AB} = \text{Head} - \text{Tail}$$

$$\overline{AB} = (1, 8, -7) - (1, 9, -6)$$

$$\overline{AB} = \langle 0 \quad -1 \quad -1 \rangle \rightarrow \text{Store to VctA}$$

$$d = |\overline{AB}| = \text{abs}(\text{VctA}) = 1.414$$

2. Find the angle between the lines AB and CD. A (-3, 2, 4), B (2, 5, -2) and C (1, -2, 2), D (4, 2, 3).

Solution:

The angle between lines AB and CD is also the same angle between the vector created by line AB and line CD.

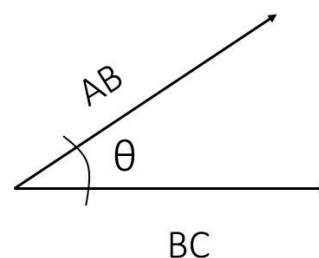
Determining the vector created by vectors AB and CD:

$$\overline{AB} = (2, 5, -2) - (-3, 2, 4)$$

$$\overline{AB} = \langle 5 \quad 3 \quad -6 \rangle$$

$$\overline{CD} = (4, 2, 3) - (1, -2, 2)$$

$$\overline{CD} = \langle 3 \quad 4 \quad 1 \rangle$$



Using the definition for the dot product:

$$\overline{AB} \cdot \overline{CD} = |\overline{AB}| |\overline{CD}| \cos \theta$$

$$\langle 5 \quad 3 \quad -6 \rangle \cdot \langle 3 \quad 4 \quad 1 \rangle = |\langle 5 \quad 3 \quad -6 \rangle| |\langle 3 \quad 4 \quad 1 \rangle| \cos \theta$$

$$21 = 8.366 \cdot 5.099 \cdot \cos \theta$$

$$\theta = 60.51^\circ$$

3. Find the Direction Cosine of the line that is perpendicular to each of the two lines whose directions are $\langle 3 \quad 0 \quad 1 \rangle$ and $\langle 2 \quad -1 \quad 2 \rangle$.

Solution:

Let:

$$A = \langle 3 \quad 0 \quad 1 \rangle$$

$$B = \langle 2 \quad -1 \quad 2 \rangle$$

Since by definition, the cross product of A and B ($A \times B$) is a vector perpendicular to A and B.

Hence,

$$\langle 3 \quad 0 \quad 1 \rangle \times \langle 2 \quad -1 \quad 2 \rangle = \langle 1 \quad -4 \quad -3 \rangle$$

And then by definition of direction cosine,

$$DC = \frac{\langle 1 \quad -4 \quad -3 \rangle}{|\langle 1 \quad -4 \quad -3 \rangle|} = \langle 0.196 \quad -0.784 \quad -0.588 \rangle$$

4. Determine the value of k and q so that ABC are collinear $A(1, 0, -2)$, $B(3, -1, 1)$ and $C(k, -3, q)$ are collinear.

Solution:

For the points ABC to be collinear, point C must be a point in the line defined by points AB ;

The equation of a line in space has a property,

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Therefore, solving for the equation of a line that defines AB :

$$\begin{aligned} &A(1, 0, -2), B(3, -1, 1) \\ \frac{x - 1}{3 - 1} &= \frac{y - 0}{-1 - 0} = \frac{z - (-2)}{1 - (-2)} \\ \frac{x - 1}{2} &= \frac{y}{-1} = \frac{z + 2}{3} \rightarrow \text{equation of line } AB \end{aligned}$$

Since point C , is a point on line AB , $C(k, -3, q)$

$$\begin{aligned} \frac{k - 1}{2} &= \frac{-3}{-1} = \frac{q + 2}{3} \\ \mathbf{k = 7; q = 7} \end{aligned}$$

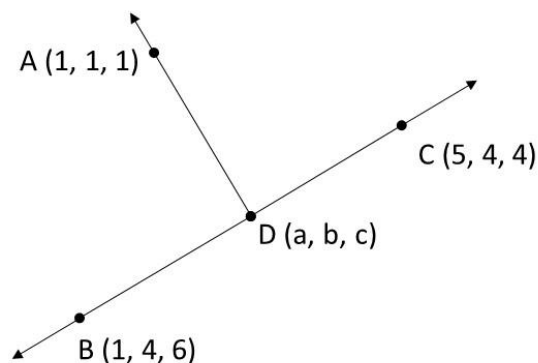
5. Find the coordinates of the foot of the perpendicular from $A(1, 1, 1)$ on the line joining $B(1, 4, 6)$ and $C(5, 4, 4)$.

Solution:

Let: Point $D(a, b, c)$ is the foot of the perpendicular from A :

Solving for the equation of line BC :

$$\begin{aligned} &B(1, 4, 6), C(5, 4, 4) \\ \frac{x - x_1}{x_2 - x_1} &= \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \\ \frac{x - 1}{5 - 1} &= \frac{y - 4}{4 - 4} = \frac{z - 6}{4 - 6} \\ \frac{x - 1}{4} &= \frac{y - 4}{0} = \frac{z - 6}{-2} \end{aligned}$$



Since point D is on line BC :

Hence substitute $D(a, b, c)$ to the equation of line BC :

$$\frac{a - 1}{4} = \frac{b - 4}{0} = \frac{c - 6}{-2}$$

Take note that the y -coordinate of B and C are equal, therefore, by principle, all y -coordinate of any point on line BC is 4 , hence $\mathbf{b = 4}$.

So, we can only use

$$\begin{aligned} \frac{a - 1}{4} &= \frac{c - 6}{-2} \\ -2a - 4c &= -26 \rightarrow \text{equation 1} \end{aligned}$$

Since BC and DA are perpendicular:

$$\overline{BC} \cdot \overline{DA} = 0$$

$$\begin{aligned} \overline{BC} &= (5, 4, 4) - (1, 4, 6) \\ \overline{BC} &= \langle 4 \quad 0 \quad -2 \rangle \\ \overline{DA} &= (1, 1, 1) - (a, b, c) \\ \overline{DA} &= \langle 1 - a \quad 1 - b \quad 1 - c \rangle \end{aligned}$$

$$\begin{aligned} \langle 4 \ 0 \ -2 \rangle \cdot \langle 1 - a \ 1 - b \ 1 - c \rangle &= 0 \\ 4(1 - a) + 0(1 - b) + (-2)(1 - c) &= 0 \\ 4 - 4a + 2c - 2 &= 0 \\ -4a + 2c &= -2 \rightarrow \text{equation 2} \end{aligned}$$

Solving equations 1 and 2, simultaneously:

$$a = 3; c = 5$$

Therefore:

$$\mathbf{D(3, 4, 5)}$$

6. Find the coordinates of the foot of the perpendicular drawn from the point (3,4,5) to the x-axis.

Solution:

Let \overline{AB} be any vector on x-axis so, we let $\langle 1 \ 0 \ 0 \rangle$ be the vector \overline{AB} for simplicity.

$$\overline{AB} = \langle 1 \ 0 \ 0 \rangle$$

And

$$\overline{AC} = (3,4,5) - (x, 0, 0)$$

$$\overline{AC} = \langle 3 - x \ 4 \ 5 \rangle$$

Since \overline{AC} and \overline{AB} are perpendicular,

Hence,

$$\overline{AC} \cdot \overline{AB} = 0$$

$$\langle 3 - x \ 4 \ 5 \rangle \cdot \langle 1 \ 0 \ 0 \rangle = 0$$

$$(3 - x)(1) + 4(0) + 5(0) = 0$$

$$3 - x = 0$$

$$x = 3$$

$$\therefore \mathbf{(3, 0, 0)}$$

7. Find the coordinates of the point where the segment joining the points (2, -2, 1) and (5, 1, -2) crosses the xy-plane.

Solution:

Let: A (2, -2, 1), B (5, 1, -2)

The point where line AB crosses the xy plane is a point on line AB such that its z-coordinate is zero.

Let that point be C (a, b, 0).

Determining the equation of line AB:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$$A(2, -2, 1), B(5, 1, -2)$$

$$\frac{x - 2}{5 - 2} = \frac{y - (-2)}{1 - (-2)} = \frac{z - 1}{-2 - 1}$$

$$\frac{x - 2}{3} = \frac{y + 2}{3} = \frac{z - 1}{-3}$$

Since C is a point on AB; thus, substitute C (a, b, 0) to the equation of the line:

$$\frac{a - 2}{3} = \frac{b + 2}{3} = \frac{0 - 1}{-3}$$

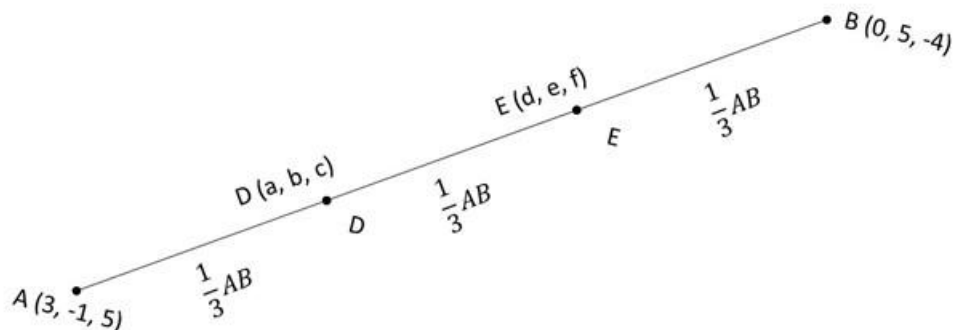
$$\frac{a - 2}{3} = \frac{1}{3}; a = 3$$

$$\frac{b + 2}{3} = \frac{1}{3}; b = -1$$

Hence the point is **(3, -1, 0)**.

8. Find the points that trisect the segment joining the points (3, -1, 5) and (0, 5, -4).

Solution:



$$D = A + \frac{1}{3}AB$$

$$E = A + \frac{2}{3}AB$$

From point D:

$$a = x_1 + \frac{1}{3}(x_2 - x_1); a = 3 + \frac{1}{3}(0 - 3) = 2$$

$$b = y_1 + \frac{1}{3}(y_2 - y_1); b = -1 + \frac{1}{3}(5 - (-11)) = 1$$

$$c = z_1 + \frac{1}{3}(z_2 - z_1); c = 5 + \frac{1}{3}(-4 - 5) = 2$$

Hence **D (2, 1, 2)**.

From point E:

$$d = x_1 + \frac{2}{3}(x_2 - x_1); d = 3 + \frac{2}{3}(0 - 3) = 1$$

$$e = y_1 + \frac{2}{3}(y_2 - y_1); e = -1 + \frac{2}{3}(5 - (-11)) = 3$$

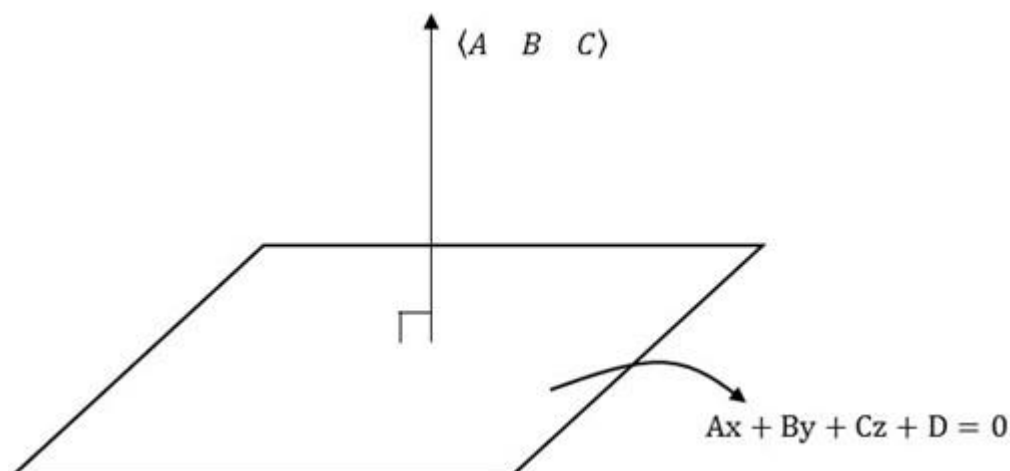
$$f = z_1 + \frac{2}{3}(z_2 - z_1); f = 5 + \frac{2}{3}(-4 - 5) = -1$$

Hence **E (1, 3, -1)**.

9. Find the equation of the plane through P (4, 3, 6) and perpendicular to the line joining P (4, 3, 6) to the point Q (2, 3, 1).

Solution:

From the property of plane:



The vector defined by $\langle A \ B \ C \rangle$ is perpendicular to the plane \overline{PQ} is a vector perpendicular to plane; hence, $\langle 2 \ 0 \ -5 \rangle$ is the coefficient of the plane:

$$\begin{aligned} -2x + 0y - 5z + D &= 0 \\ -2x - 5z + D &= 0 \end{aligned}$$

But since P is on the plane, thus substitute P (4, 3, 6):

$$\begin{aligned} -2(4) - 5(6) + D &= 0 \\ D &= 38 \\ \mathbf{2x + 5z - 38 = 0} \end{aligned}$$

10. Find the equation of the plane through the point P (1, 2, -1) and parallel to the plane $2x - 3y + 4z + 6 = 0$.

Solution:

For the equation of plane to be parallel, the coefficients must be equal this, for any two plane:

$$\begin{aligned} Ax + By + Cz + D_1 &= 0 \\ Ax + By + Cz + D_2 &= 0 \end{aligned}$$

Plane 1: $2x - 3y + 4z + 6 = 0$

Plane 2: $2x - 3y + 4z + D = 0$

Substitute (1, 2, -1) to the equation to obtain the value of D:

$$\begin{aligned} 2(1) - 3(2) + 4(-1) + D &= 0 \\ D &= 8 \\ \mathbf{2x - 3y + 4z + 8 = 0} \end{aligned}$$

11. Find the equation of the plane that contains the three points P (1, -2, 4), Q (4, 1, 7), R (-1, 5, 1).

Solution:

The general equation of a plane is:

$$Ax + By + Cz + D = 0$$

Dividing the equation by D:

$$\frac{A}{D}x + \frac{B}{D}y + \frac{C}{D}z = 1$$

Take note that the variables A, B, C, and D could be any variable.

Substituting points P, Q, and R to the equation:

$$\frac{A}{D}(1) + \frac{B}{D}(-2) + \frac{C}{D}(4) = 1$$

$$\frac{A}{D}(4) + \frac{B}{D}(1) + \frac{C}{D}(7) = 1$$

$$\frac{A}{D}(-1) + \frac{B}{D}(5) + \frac{C}{D}(1) = 1$$

Solving for the value of unknowns, simultaneously:

$$\frac{A}{D} = -\frac{5}{12}$$

$$\frac{B}{D} = \frac{1}{24}$$

$$\frac{C}{D} = \frac{3}{8}$$

Substitute:

$$-\frac{5}{12}x + \frac{1}{24}y + \frac{3}{8}z = 1$$

$$10x - y - 9z = -24$$

12. Find the perpendicular distance between the point $(-2, 8, -3)$ and plane $9x - y - 4z + 1 = 0$.

Solution:

The distance d from plane $Ax + By + Cz + D = 0$ to the point (x_1, y_1, z_1)

$$d = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}}$$

Where the sign of the radical is taken opposite to that of D

The numerical value of the result is the distance of point to plane, if the sign is negative it signifies that the given point and the origin are on the same side of the plane, and if positive means the point is not on the same side as the origin.

The distance between a plane $Ax + By + Cz + D = 0$ and a point (x_1, y_1, z_1) :

$$d = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}}$$

$$d = \frac{9(-2) + (-1)(8) + (-4)(-3) + 1}{-\sqrt{9^2 + (-1)^2 + (-4)^2}} \approx 1.313$$

13. Find the perpendicular distance between the two planes $x - 2y + 2z = 6$, $3x - 6y + 6z = 2$.

Solution:

Distance between two parallel plane:

$$d = \left| \frac{D_1 - D_2}{\sqrt{A^2 + B^2 + C^2}} \right|$$

Plane 1: $x - 2y + 2z - 6 = 0$

Plane 2: $3x - 6y + 6z - 2 = 0 \rightarrow x - 2y + 2z - \frac{2}{3} = 0$

$$d = \left| \frac{-6 - \left(-\frac{2}{3}\right)}{\sqrt{1^2 + (-2)^2 + 2^2}} \right| = \frac{16}{9} \approx 1.778$$

14. Find the equation of the line through the following points (1,3, -2), (2,2,0), in symmetric form.

Solution:

The equation of a line in symmetric form is:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$$P_1(1,3,-2), P_2(2,2,0)$$

$$\frac{x - 1}{2 - 1} = \frac{y - 3}{2 - 3} = \frac{z - (-2)}{0 - (-2)}$$

$$\frac{x - 1}{1} = \frac{y - 3}{-1} = \frac{z + 2}{2}$$

15. Find the equation of the line through the following points (1,3, -2), (2,2,0), in parametric form.

Solution:

To transform the equation in parametric form:

Let:

$$t = \frac{x - 1}{1} = \frac{y - 3}{-1} = \frac{z + 2}{2}$$

$$\left. \begin{array}{l} \frac{x - 1}{1} = t \mid x = t + 1 \\ \frac{y - 3}{-1} = t \mid y = -t + 3 \\ \frac{z + 2}{2} = t \mid z = 2t - 2 \end{array} \right\} \rightarrow \text{parametric form}$$

16. Find the point on the line $x = y = z$, that is equidistant from the points (3, 0, 5) and (1, -1, 4).

Solution:

Let P(a, b, c) be a point equidistant from point A(3, 0, 5) and B(1, -1, 4).

$$d_{PA} = d_{PB}$$

Since $x = y = z$; hence, $a = b = c$:

Thus,

$$a = b$$

$$a = c$$

$$\left(\sqrt{(a - 3)^2 + (b - 0)^2 + (c - 5)^2} \right)^2 = \left(\sqrt{(a - 1)^2 + (b + 1)^2 + (c - 4)^2} \right)^2$$

$$(a - 3)^2 + (b - 0)^2 + (c - 5)^2 = (a - 1)^2 + (b + 1)^2 + (c - 4)^2$$

Substituting $a = b, a = c$:

$$(a - 3)^2 + a^2 + (a - 5)^2 = (a - 1)^2 + (a + 1)^2 + (a - 4)^2$$

$$a^2 - 6a + 9 + a^2 + a^2 - 10a + 25$$

$$= a^2 - 2a + 1 + a^2 + 2a + a + a^2 - 8a + 16$$

$$-16a + 34 = -8a + 18$$

$$a = 2$$

Hence: **P(2, 2, 2)**.

17. Find the area of the triangle whose sides have the equations

$$\frac{x}{3} = \frac{y}{4} = \frac{z}{5}; \frac{x}{2} = \frac{y}{1} = \frac{z}{-2}; \frac{x}{2} = \frac{y+5}{3} = \frac{z+16}{7}$$

Solution:

Let:

$$L_1: \frac{x}{3} = \frac{y}{4} = \frac{z}{5}$$

$$L_2: \frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

$$L_3: \frac{x}{2} = \frac{y+5}{3} = \frac{z+16}{7}$$

Intersection of L_1 and L_2 :

$$L_1: \frac{x}{3} = \frac{y}{4} = \frac{z}{5}; L_2: \frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

Equating:

$$\frac{x}{3} = \frac{y}{4} \rightarrow 4x - 3y = 0$$

$$\frac{x}{2} = \frac{y}{1} \rightarrow x - 2y = 0$$

Hence,

$$x = 0, y = 0$$

To solve for z use any equation, say L_1 :

$$\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$$

$$\frac{0}{3} = \frac{0}{4} = \frac{z}{5}$$

$$z = 0$$

Thus, the intersection of L_1 and L_2 is $(0, 0, 0)$.

Intersection of L_1 and L_3 :

$$L_1: \frac{x}{3} = \frac{y}{4} = \frac{z}{5}; L_3: \frac{x}{2} = \frac{y+5}{3} = \frac{z+16}{7}$$

Equating:

$$\frac{x}{3} = \frac{y}{4} \rightarrow 4x - 3y = 0$$

$$\frac{x}{2} = \frac{y+5}{3} \rightarrow 3x - 2y = 10$$

Hence,

$$x = 30, y = 40$$

To solve for z use any equation, say L_1 :

$$\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$$

$$\frac{30}{3} = \frac{40}{4} = \frac{z}{5}$$

$$z = 50$$

Thus, the intersection of L_1 and L_3 is $(30, 40, 50)$.

Intersection of L_2 and L_3 :

$$L_2: \frac{x}{2} = \frac{y}{1} = \frac{z}{-2}; L_3: \frac{x}{2} = \frac{y+5}{3} = \frac{z+16}{7}$$

Equating:

$$\frac{x}{2} = \frac{y}{1} \rightarrow x - 2y = 0$$

$$\frac{x}{2} = \frac{y+5}{3} \rightarrow 3x - 2y = 10$$

Hence,

$$x = 5, y = \frac{5}{2}$$

To solve for z use any equation, say L_2 :

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

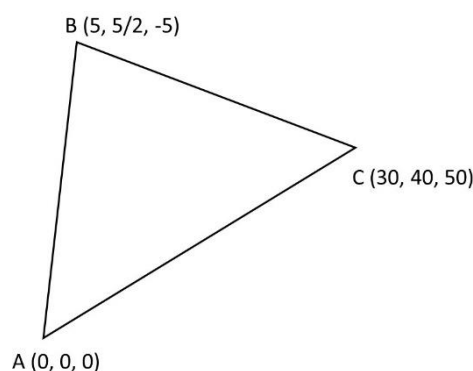
$$\frac{5}{2} = \frac{5/2}{1} = \frac{z}{-2}$$

$$z = -5$$

Thus, the intersection of L_1 and L_3 is $(5, \frac{5}{2}, -5)$.

The points of intersection of lines:

$$A(0, 0, 0), B(30, 40, 50), C(5, \frac{5}{2}, -5)$$



$$A_T = \frac{1}{2} |AB \times AC|$$

$$\overline{AB} = (30, 40, 50) - (0, 0, 0)$$

$$\overline{AB} = \langle 30 \quad 40 \quad 50 \rangle$$

$$\overline{AC} = (5, \frac{5}{2}, -5) - (0, 0, 0)$$

$$\overline{AC} = \langle 5 \quad \frac{5}{2} \quad -5 \rangle$$

$$A_T = \frac{1}{2} \left| \langle 30 \quad 40 \quad 50 \rangle \times \langle 5 \quad \frac{5}{2} \quad -5 \rangle \right|$$

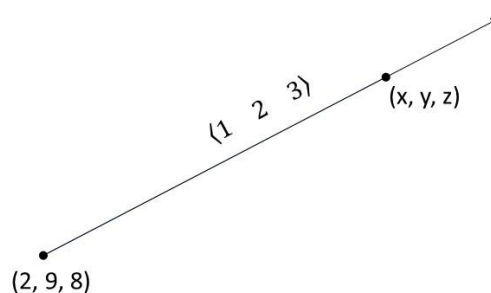
$$A_T = 251.95$$

18. Find the equation of a line formed if Vector $A = \langle 1 \quad 2 \quad 3 \rangle$ is made to pass at point $(2, 9, 8)$.

Solution:

By inspection:

$$\frac{x-2}{1} = \frac{y-9}{2} = \frac{z-8}{3}$$



19. Find the center and radius of the sphere whose equation is $x^2 + y^2 + z^2 = 6x + 8z$.

Solution:

The center-radius form of sphere with center (h, k, t) and radius r is:

$$(x - h)^2 + (y - k)^2 + (z - t)^2 = r^2$$

By completing the square:

$$\begin{aligned} x^2 - 6x + y^2 + z^2 - 8z &= 0 \\ x^2 - 6x + 9 + y^2 + z^2 - 8z + 16 &= 9 + 16 \\ (x - 3)^2 + y^2 + (z - 4)^2 &= 5^2 \end{aligned}$$

$$\therefore \mathbf{C(3, 0, 4); r = 5}$$

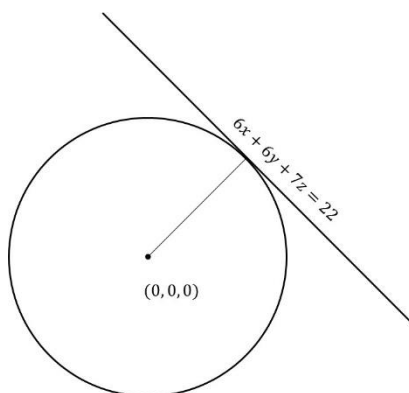
20. Find the equation of the sphere tangent to the plane $6x + 6y + 7z = 22$, and has its center at origin and point of tangency.

Solution:

r = distance of the plane to origin

$$\begin{aligned} d &= \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}} \\ d &= \frac{6(0) + 6(0) + 7(0) - 22}{\pm\sqrt{6^2 + 6^2 + 7^2}} = 2 \end{aligned}$$

$$\therefore \mathbf{x^2 + y^2 + z^2 = 4}$$



VECTORS

Situation: Determine the absolute value of the following vectors:

1. $A = 2i + 3j$

A. 3.606	C. 5.342
B. 6.403	D. 7.348

2. $B = 2i + 6j + k$

A. 3.606	C. 5.342
B. 6.403	D. 7.348

3. $C = \langle 2 \ 7 \ 1 \rangle$

A. 3.606	C. 5.342
B. 6.403	D. 7.348

Situation: Let $A = \langle 2 \ -3 \ 6 \rangle$, $B = \langle 1 \ 8 \ 4 \rangle$, determine:

4. $A + B$

A. $\langle -3 \ 5 \ 10 \rangle$	C. $\langle 3 \ -5 \ 10 \rangle$
B. $\langle 3 \ 5 \ 10 \rangle$	D. $\langle 3 \ 5 \ -10 \rangle$

5. $|A + B|$

A. 11.576	C. 12.511
B. 13.451	D. 10.144

6. The x-component of resultant of A and B:

A. 3	C. 2
B. 5	D. 10

7. The y-component of resultant of A and B:

A. 3	C. 2
B. 5	D. 10

8. The z-component of resultant of A and B:

A. 3	C. 2
B. 5	D. 10

9. $A \cdot B$

A. 3	C. 2
B. 5	D. 10

10. $A \times B$

A. $\langle 60 \ -2 \ 19 \rangle$	C. $\langle -60 \ 2 \ 19 \rangle$
B. $\langle -60 \ -2 \ -19 \rangle$	D. $\langle -60 \ -2 \ 19 \rangle$

11. $|A \times B|$

A. 62.968	C. 65.415
B. 60.516	D. 67.348

12. The unit vector of A:

- A. $\langle 0.286 \quad -0.429 \quad 0.857 \rangle$ C. $\langle 0.111 \quad 0.556 \quad 0.889 \rangle$
 B. $\langle 0.111 \quad 0.889 \quad 0.444 \rangle$ D. $\langle 0.012 \quad 0.123 \quad 0.666 \rangle$

13. The unit vector of B:

- A. $\langle 0.286 \quad -0.429 \quad 0.857 \rangle$ C. $\langle 0.111 \quad 0.556 \quad 0.889 \rangle$
 B. $\langle 0.111 \quad 0.889 \quad 0.444 \rangle$ D. $\langle 0.012 \quad 0.123 \quad 0.666 \rangle$

14. The direction cosine of A:

- A. $\langle 0.111 \quad 0.556 \quad 0.889 \rangle$ C. $\langle 0.286 \quad -0.429 \quad 0.857 \rangle$
 B. $\langle 0.012 \quad 0.123 \quad 0.666 \rangle$ D. $\langle 0.111 \quad 0.889 \quad 0.444 \rangle$

15. The direction cosine of B:

- A. $\langle 0.111 \quad 0.556 \quad 0.889 \rangle$ C. $\langle 0.286 \quad -0.429 \quad 0.857 \rangle$
 B. $\langle 0.012 \quad 0.123 \quad 0.666 \rangle$ D. $\langle 0.111 \quad 0.889 \quad 0.444 \rangle$

Situation: Let $A = \langle 1 \quad 2 \quad 1 \rangle$, $B = \langle 2 \quad 0 \quad -1 \rangle$, and $C = \langle 0 \quad -1 \quad 2 \rangle$, evaluate:

16. $(A \times B) \cdot C$

- A. -11 C. 12
 B. 11 D. -12

17. $A \times (B \times C)$

- A. $\langle 0 \quad 1 \quad -2 \rangle$ C. $\langle -22 \quad 0 \quad 11 \rangle$
 B. $\langle 4 \quad -6 \quad 8 \rangle$ D. $\langle 2 \quad 4 \quad 2 \rangle$

18. $(A \times B) \times C$

- A. $\langle 0 \quad 1 \quad -2 \rangle$ C. $\langle -22 \quad 0 \quad 11 \rangle$
 B. $\langle 4 \quad -6 \quad 8 \rangle$ D. $\langle 2 \quad 4 \quad 2 \rangle$

19. $(A \times B) \times (B \times C)$

- A. $\langle 0 \quad 1 \quad -2 \rangle$ C. $\langle -22 \quad 0 \quad 11 \rangle$
 B. $\langle 4 \quad -6 \quad 8 \rangle$ D. $\langle 2 \quad 4 \quad 2 \rangle$

20. $(B \cdot C) \times (A \times B)$

- A. $\langle 0 \quad 1 \quad -2 \rangle$ C. $\langle -22 \quad 0 \quad 11 \rangle$
 B. $\langle 4 \quad -6 \quad 8 \rangle$ D. $\langle 2 \quad 4 \quad 2 \rangle$

21. A^2

- A. 3 C. 4
 B. 5 D. 6

22. B^2

- A. 3 C. 4
 B. 5 D. 6

23. Which of the following is equal to $\frac{1}{4}(|A + B|^2 - |A - B|^2)$?

- A. $A \cdot B$ C. $A \times B$
 B. $4A \cdot B$ D. $4A \times B$

Situation: A concurrent force system in space is composed of 3 forces described as follows. A = 100 kN acts through point (2, 6, 8) and (1, 4, 2). B = 330 kN and acts through the point (2, 6, 8) and (4, 1, -2). C = 85 kN and acts through the point (2,6,8) and (2, -3, 3).

34. What is the x-component of the resultant force (kN)?

- A. -425.533
- B. -250.813
- C. 42.492
- D. 495.773

35. What is the y-component of the resultant force (kN)?

- A. -425.533
- B. -250.813
- C. 42.492
- D. 495.773

36. What is the z-component of the resultant force (kN)?

- A. -425.533
- B. -250.813
- C. 42.492
- D. 495.773

37. What is the resultant force (kN)?

- A. -425.533
- B. -250.813
- C. 42.492
- D. 495.773

VECTORS SOLUTIONS

Situation: Determine the absolute value of the following vectors:

1. $A = 2i + 3j$

Solution:

$$|A| = \sqrt{2^2 + 3^2} = \sqrt{13} \approx 3.606$$

Or using calculator:

Mode - 8 (Vector):

Select any vector, say 1: VctA, so press 1 to select VctA, since the vector has two dimensions, press 2:

And then input:

$$[2 \quad 3]$$

And then press AC.

Then press:

Shift - hyp - Shift - 5 - 3 -)

$$\mathbf{Abs(VctA) = 3.606}$$

2. $B = 2i + 6j + k$

Solution:

Store $\langle 2 \quad 6 \quad 1 \rangle$ to VctB and then press $\mathbf{Abs(VctB) = 6.403}$

3. $C = \langle 2 \quad 7 \quad 1 \rangle$

Solution:

Store $\langle 2 \quad 7 \quad 1 \rangle$ to Vct C and then press $\mathbf{Abs(VctC) = 7.348}$

Situation: Let $A = \langle 2 \quad -3 \quad 6 \rangle$, $B = \langle 1 \quad 8 \quad 4 \rangle$, determine:

4. $A + B$

Solution:

When adding vectors, add its corresponding components:

$$A + B = A = \langle 2 \quad -3 \quad 6 \rangle + \langle 1 \quad 8 \quad 4 \rangle = \langle 3 \quad 5 \quad 10 \rangle$$

5. $|A + B|$

Solution:

$$|A + B| = |\langle 3 \quad 5 \quad 10 \rangle| = \sqrt{3^2 + 5^2 + 10^2} = \sqrt{134} \approx 11.576$$

6. The x-component of resultant of A and B:

Solution:

The components of a vector are expressed in the form $\langle x \quad y \quad z \rangle$

Hence, the x-component of the resultant of A and B is 3.

7. The y-component of resultant of A and B:

Solution:

The components of a vector are expressed in the form $\langle x \quad y \quad z \rangle$

Hence, the y-component of the resultant of A and B is 5.

8. The z-component of resultant of A and B:

Solution:

The components of a vector are expressed in the form $\langle x \ y \ z \rangle$

Hence, the z-component of the resultant of A and B is **10**.

9. $A \cdot B$

Solution:

If $A = \langle x_1 \ y_1 \ z_1 \rangle$ and $B = \langle x_2 \ y_2 \ z_2 \rangle$, then,

$$A \cdot B = x_1x_2 + y_1y_2 + z_1z_2$$

Therefore,

$$A \cdot B = \langle 2 \ -3 \ 6 \rangle \cdot \langle 1 \ 8 \ 4 \rangle$$

$$A \cdot B = 2(1) + (-3)(8) + 6(4) = 2$$

Another solution:

Using your calculator, press:

Shift – 5 – 3 – Shift – 5 – 7 – Shift – 5 – 4

$$\mathbf{VctA \cdot VctB = 2}$$

Take note that dot product is commutative, thus,

$$VctA \cdot VctB = VctB \cdot VctA$$

10. $A \times B$

Solution:

$$A = A_1i + A_2j + A_3k; B = B_1i + B_2j + B_3k$$

$$A \times B = (A_1B_3 - A_3B_2)i + (A_3B_1 - A_1B_3)j + (A_1B_2 - A_2B_1)k$$

Or in determinant:

$$A \times B = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Therefore,

$$A \times B = [(-3)(4) - (6)(8)]i + [(6)(1) - (2)(4)]j + [(2)(8) - (-3)(1)]k$$

$$\mathbf{A \times B = -60i - 2j + 19k}$$

Another solution:

Using your calculator, press:

Shift – 5 – 3 – × – Shift – 5 – 4

$$\mathbf{VctA \times VctB = \langle -60 \ -2 \ 19 \rangle}$$

11. $|A \times B|$

Solution:

$$|A \times B| = \sqrt{(-60)^2 + (-2)^2 + 19^2} = \mathbf{62.968}$$

Another solution:

Using your calculator, press:

Shift – hyp – Shift – 5 – 3 – × – Shift – 5 – 4

$$\mathbf{Abs(VctA \times VctB) = 62.968}$$

12. The unit vector of A:

Solution:

Unit vector (λ) of vector (V) is a vector whose orientation is the same as vector (V) and whose magnitude is 1.

Hence,

$$\lambda = \frac{V}{|V|}$$

$$\lambda_A = \frac{A}{|A|} = \frac{1}{|A|} \cdot A$$

$$|A| = 7$$

$$\lambda_A = \frac{1}{7} \langle 2 \quad -3 \quad 6 \rangle = \langle 2/7 \quad -3/7 \quad 6/7 \rangle = \langle 0.286 \quad -0.429 \quad 0.857 \rangle$$

Another solution:

Using your calculator, press:

Shift – 5 – 3 – ÷ – Shift – hyp – Shift – 5 – 3

$$\mathbf{VctA} \div \mathbf{Abs(VctA)} = \langle 0.286 \quad -0.429 \quad 0.857 \rangle$$

13. The unit vector of B:

Solution:

$$\lambda_B = \frac{B}{|B|} = \frac{1}{|B|} \cdot B$$

$$|B| = 9$$

$$\lambda_B = \frac{1}{9} \langle 1 \quad 8 \quad 4 \rangle = \langle 1/9 \quad -8/9 \quad 4/9 \rangle = \langle 0.111 \quad 0.889 \quad 0.444 \rangle$$

Another solution:

Using your calculator, press:

Shift – 5 – 4 – ÷ – Shift – hyp – Shift – 5 – 4

$$\mathbf{VctB} \div \mathbf{Abs(VctB)} = \langle 0.111 \quad 0.889 \quad 0.444 \rangle$$

14. The direction cosine of A:

Solution:

The direction cosine of a vector, $\bar{V} = \langle \cos \theta_x \quad \cos \theta_y \quad \cos \theta_z \rangle$ is equal to the unit vector of \bar{V} , $\lambda = \langle \lambda_x \quad \lambda_y \quad \lambda_z \rangle$

Hence,

$$\langle \cos \theta_x \quad \cos \theta_y \quad \cos \theta_z \rangle = \langle \lambda_x \quad \lambda_y \quad \lambda_z \rangle$$

Therefore, the direction cosine of vector A is simply

$$\langle 0.286 \quad -0.429 \quad 0.857 \rangle.$$

15. The direction cosine of B:

Solution:

The direction cosine of vector A is simply $\langle 0.111 \quad 0.889 \quad 0.444 \rangle$.

Situation: Let $A = \langle 1 \ 2 \ 1 \rangle$, $B = \langle 2 \ 0 \ -1 \rangle$, and $C = \langle 0 \ -1 \ 2 \rangle$, evaluate:

16. $(A \times B) \cdot C$

Solution:

First, store the given vectors to Vectors A, B, and C, respectively.

Then type:

$$(-\text{Shift} - 5 - 3 - \times - \text{Shift} - 5 - 4 -) - \text{Shift} - 5 - 7 - \text{Shift} - 5 - 5 \\ (\mathbf{VctA} \times \mathbf{VctB}) \cdot \mathbf{VctC} = -11$$

17. $A \times (B \times C)$

Solution:

Take note: Cross product is non-commutative. Therefore, you must follow the proper order of the vectors.

$$\text{Shift} - 5 - 3 - \times - (-\text{Shift} - 5 - 4 - \times - \text{Shift} - 5 - 5 -) \\ \mathbf{VctA} \times (\mathbf{VctB} \times \mathbf{VctC}) = \langle 0 \ 1 \ -2 \rangle$$

18. $(A \times B) \times C$

Solution:

$$(-\text{Shift} - 5 - 3 - \times - \text{Shift} - 5 - 4 -) - \times - \text{Shift} - 5 - 5 \\ (\mathbf{VctA} \times \mathbf{VctB}) \times \mathbf{VctC} = \langle 2 \ 4 \ 2 \rangle$$

19. $(A \times B) \times (B \times C)$

Solution:

$$(-\text{Shift} - 5 - 3 - \times - \text{Shift} - 5 - 4 -) - \times - (-\text{Shift} - 5 - 4 - \times - \text{Shift} - 5 \\ - 5 -) \\ (\mathbf{VctA} \times \mathbf{VctB}) \times (\mathbf{VctB} \times \mathbf{VctC}) = \langle -22 \ 0 \ 11 \rangle$$

20. $(B \cdot C) \times (A \times B)$

Solution:

$$(-\text{Shift} - 5 - 4 - \text{Shift} - 5 - 7 - \text{Shift} - 5 - 5 -) \\ - \times - (-\text{Shift} - 5 - 3 - \times - \text{Shift} - 5 - 4 -) \\ (\mathbf{VctB} \cdot \mathbf{VctC}) \times (\mathbf{VctA} \times \mathbf{VctB}) = \langle 4 \ -6 \ 8 \rangle$$

21. A^2

Solution:

Note:

The square of a vector is equal to the square of its absolute value. Hence,

$$A^2 = A \cdot A = |A|^2$$

Therefore:

$$A^2 = (\text{Abs}(\mathbf{VctA}))^2 = 6$$

22. B^2

Solution:

$$B^2 = (\text{Abs}(\mathbf{VctB}))^2 = 5$$

23. Which of the following is equal to $\frac{1}{4}(|A + B|^2 - |A - B|^2)$?

Solution:

$$\begin{aligned} \frac{1}{4}(|A + B|^2 - |A - B|^2) &= \frac{1}{4}[(A + B)^2 - (A - B)^2] \\ \frac{1}{4}(|A + B|^2 - |A - B|^2) &= \frac{1}{4}[(A + B) \cdot (A + B) - (A - B) \cdot (A - B)] \\ \frac{1}{4}(|A + B|^2 - |A - B|^2) &= \frac{1}{4}\{[A \cdot (A + B) + B \cdot (A + B)] - [A \cdot (A - B) - B \cdot (A - B)]\} \\ \frac{1}{4}(|A + B|^2 - |A - B|^2) &= \frac{1}{4}[(A \cdot A + A \cdot B + B \cdot A + B \cdot B) - (A \cdot A - A \cdot B - B \cdot A + B \cdot B)] \\ \frac{1}{4}(|A + B|^2 - |A - B|^2) &= \frac{1}{4}[4A \cdot B] = \mathbf{A \cdot B} \end{aligned}$$

24. Let $A = \langle 2 \quad -3 \quad 6 \rangle$ and $B = \langle 1 \quad 8 \quad -4 \rangle$. Determine the angle between vector A and B.

Solution:

From the definition of dot product:

$$A \cdot B = |A||B| \cos \theta$$

Therefore,

$$\cos \theta = \frac{A \cdot B}{|A||B|}$$

Store the given vectors into vectors A and B, respectively.

To get the angle between the vectors, just press:

$$\cos^{-1} \left(\frac{\text{VctA} \cdot \text{VctB}}{(\text{Abs}(\text{VctA}) \times \text{Abs}(\text{VctB}))} \right) = \mathbf{136.90^\circ}$$

25. Find a unit vector parallel to the sum of the vectors $A = \langle 2 \quad 4 \quad -5 \rangle$ and $B = \langle 1 \quad 2 \quad 3 \rangle$.

Solution:

The unit vector parallel to the sum of A and B is a vector whose orientation or direction in space is equal to $A + B$, and has a magnitude of 1 or simply, that is the unit vector of $A + B$.

Solving for $A + B$:

Store the given vectors into vectors A and B, respectively.

Hence,

$$\begin{aligned} \lambda &= \frac{A + B}{|A + B|} \\ \lambda &= \frac{\text{VctA} + \text{VctB}}{\text{Abs}(\text{VctA} + \text{VctB})} \\ \lambda &= (\text{VctA} + \text{VctB}) \div \text{Abs}(\text{VctA} + \text{VctB}) \\ \lambda &= \langle \mathbf{0.429} \quad \mathbf{0.857} \quad \mathbf{-0.286} \rangle \end{aligned}$$

26. Find the value of m such that $A = \langle m \ -2 \ 1 \rangle$ and $B = \langle 2m \ m \ -4 \rangle$ are perpendicular.

Solution:

From the definition of dot product:

$$A \cdot B = |A||B| \cos \theta$$

If $\theta = 90^\circ$:

$$A \cdot B = |A||B| \cos 90^\circ$$

$$A \cdot B = 0$$

$$A = \langle m \ -2 \ 1 \rangle$$

$$B = \langle 2m \ m \ -4 \rangle$$

$$\langle m \ -2 \ 1 \rangle \cdot \langle 2m \ m \ -4 \rangle = 0$$

$$m(2m) + (-2)(m) + (1)(-4) = 0$$

$$2m^2 - 2m - 4 = 0$$

$$m^2 - m - 2 = 0$$

Hence,

$$\mathbf{m = 2, m = -1}$$

27. Find a unit vector parallel to xy -plane and perpendicular to the vector $\langle 4 \ -3 \ -1 \rangle$.

Solution:

Let: λ be the unit vector parallel to xy -plane, thus there is no z -component.

$$\lambda = \langle x \ y \ 0 \rangle; |\lambda| = \sqrt{x^2 + y^2} = 1$$

$$\sqrt{x^2 + y^2} = 1 \rightarrow \text{equation 1}$$

Store $\langle 4 \ -3 \ -1 \rangle$ to vector A .

$$\lambda \cdot A = 0$$

$$\langle x \ y \ 0 \rangle \cdot \langle 4 \ -3 \ -1 \rangle = 0$$

$$4x - 3y = 0$$

$$x = \frac{3}{4}y \rightarrow \text{equation 2}$$

$$\left(\frac{3}{4}y\right)^2 + y^2 = 1$$

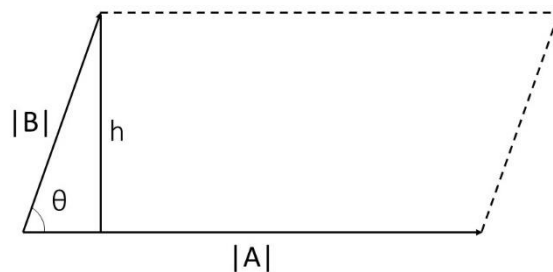
$$\frac{9}{16}y^2 + y^2 = 1$$

$$\frac{25}{16}y^2 = 1$$

$$y = \frac{4}{5}; x = \frac{3}{4}\left(\frac{4}{5}\right) = \frac{3}{5}$$

$$\therefore \lambda = \left\langle \frac{4}{5} \ \frac{3}{5} \ 0 \right\rangle$$

28. Find the area of the parallelogram if two sides are represented by vector $\langle 1 \ -2 \ 4 \rangle$ and $\langle 5 \ 7 \ 9 \rangle$.



Solution:

$$\text{Area, } A = |A|h$$

$$A = |A||B| \sin \theta$$

From the definition of cross product:

$$|A \times B| = |A||B| \sin \theta$$

Hence, the area of parallelogram is:

$$\text{Area} = |A \times B|$$

Store the given vectors into vectors A and B, respectively.

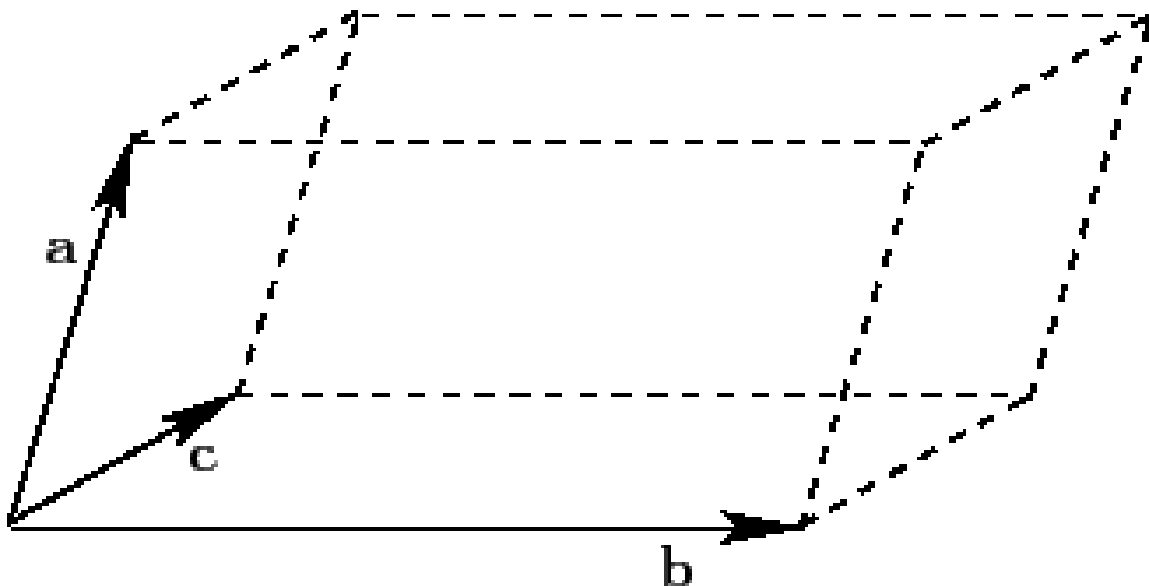
To solve for the area:

$$\text{Area} = \text{Abs}(\text{VctA} \times \text{VctB}) = \mathbf{50.259}$$

29. Find the volume of the parallelepiped if the sides are represented by vectors $\langle -1 \ 2 \ 8 \rangle$, $\langle 3 \ -4 \ 7 \rangle$ and $\langle 9 \ -2 \ 5 \rangle$.

Solution:

The volume of parallelepiped is equal to the numerical value of



$$[ABC] = (A \times B) \cdot C \text{ or}$$

$$[BCA] = (B \times C) \cdot A \text{ or}$$

$$[ACB] = (A \times C) \cdot B \text{ or}$$

$$[BCA] = (B \times C) \cdot A$$

Any arrangement will do, if ABC is there.

Store the given vectors to vectors A, B, and C, respectively.

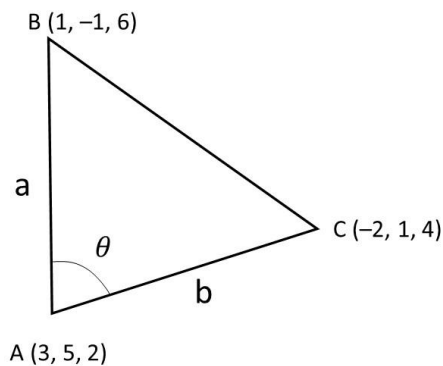
Hence,

$$V = (\text{VctA} \times \text{VctB}) \cdot \text{VctC}$$

$$V = \mathbf{342}$$

30. Find the area of the triangle with vertices $(3, 5, 2)$, $(1, -1, 6)$, and $(-2, 1, 4)$.

Solution:



To solve for the area of triangle, we can use:

$$A_T = \frac{1}{2} ab \sin \theta$$

Note: It can be any two adjacent sides and an included angle.

$$a = |AB|$$

$$b = |AC|$$

$$AB = (1, -1, 6) - (3, 5, 2) = \langle -2 \quad -6 \quad 4 \rangle \rightarrow \text{VctA}$$

$$AC = (-2, 1, 4) - (3, 5, 2) = \langle -5 \quad -4 \quad 2 \rangle \rightarrow \text{VctB}$$

$$A_T = \frac{1}{2} |AB| |AC| \sin \theta$$

$$A_T = \frac{1}{2} |AB \times AC|$$

$$A_T = \frac{1}{2} \text{Abs}(\text{VctA} \times \text{VctB})$$

$$A_T = \mathbf{13.748}$$

31. Find a unit vector perpendicular to both vectors $\langle 2 \quad 1 \quad -1 \rangle$ and $\langle 3 \quad 4 \quad -1 \rangle$.

Solution:

From the definition of cross product, the result of cross product is a vector perpendicular to both vectors A and B.

Thus, if $C = A \times B$:

$$A \cdot C = 0$$

$$B \cdot C = 0$$

$$\text{VctA} = \langle 2 \quad 1 \quad -1 \rangle$$

$$\text{VctB} = \langle 3 \quad 4 \quad -1 \rangle$$

Hence,

$$\text{VctA} \times \text{VctB} = \langle 3 \quad -1 \quad 5 \rangle$$

$$\lambda_{A \times B} = \frac{\text{VctA} \times \text{VctB}}{|\text{VctA} \times \text{VctB}|}$$

$$\lambda_{A \times B} = \frac{\langle 3 \quad -1 \quad 5 \rangle}{|\langle 3 \quad -1 \quad 5 \rangle|}$$

$$\lambda_{A \times B} = \left\langle \frac{3}{\sqrt{35}} \quad -\frac{1}{\sqrt{35}} \quad \frac{5}{\sqrt{35}} \right\rangle$$

$$\begin{aligned} \text{VctB} \times \text{VctA} &= \langle -3 \quad 1 \quad -5 \rangle \\ \lambda_{\text{B} \times \text{A}} &= \frac{\langle -3 \quad 1 \quad -5 \rangle}{|\langle -3 \quad 1 \quad -5 \rangle|} \\ \lambda_{\text{B} \times \text{A}} &= \left\langle -\frac{3}{\sqrt{35}} \quad \frac{1}{\sqrt{35}} \quad -\frac{5}{\sqrt{35}} \right\rangle \end{aligned}$$

32. Find the value of m that makes the vectors $A = \langle 1 \quad 1 \quad -1 \rangle$, $B = \langle 2 \quad -1 \quad 1 \rangle$ and $C = \langle m \quad -1 \quad m \rangle$ coplanar.

Solution:

Three vectors are said to be coplanar, if they all lie in the same plane (i.e., the vectors are on the plane)

Hence, they will not form a parallelepiped, therefore, the volume is zero.

$$(A \times B) \cdot C = 0$$

Store vectors $A = \langle 1 \quad 1 \quad -1 \rangle$ and $B = \langle 2 \quad -1 \quad 1 \rangle$ to vectors A and B , respectively.

$$\begin{aligned} (\langle 1 \quad 1 \quad -1 \rangle \times \langle 2 \quad -1 \quad 1 \rangle) \cdot \langle m \quad -1 \quad m \rangle &= 0 \\ \langle 0 \quad -3 \quad -3 \rangle \cdot \langle m \quad -1 \quad m \rangle &= 0 \\ 0(m) + (-3)(-1) + (-3)(m) &= 0 \\ 3 - 3m &= 0 \\ \mathbf{m} &= \mathbf{1} \end{aligned}$$

33. A force F of 4000 kN is passing from $A(1, 2, 7)$ to $B(-2, 3, 5)$. Find the components of the force.

Solution:

Determine the vector created by AB :

$$\begin{aligned} \overline{AB} &= \text{Head} - \text{Tail} \\ \overline{AB} &= (-2, 3, 5) - (1, 2, 7) \\ \overline{AB} &= \langle -3 \quad 1 \quad -2 \rangle \\ \lambda_{AB} &= \frac{\overline{AB}}{|\overline{AB}|} \\ \lambda_{AB} &= \frac{\langle -3 \quad 1 \quad -2 \rangle}{|\langle -3 \quad 1 \quad -2 \rangle|} \\ \lambda_{AB} &= \langle -0.8017 \quad 0.2673 \quad -0.5345 \rangle \end{aligned}$$

This vector defines the path of the force F .

To find the component of the force, multiply the force to the unit vector of AB :

$$\begin{aligned} F &= 4000 \langle -0.8017 \quad 0.2673 \quad -0.5345 \rangle \\ \mathbf{F} &= \langle -3207.13 \quad 1069.045 \quad -2138.090 \rangle \end{aligned}$$

Situation: A concurrent force system in space is composed of 3 forces described as follows. $A = 100$ kN acts through point $(2, 6, 8)$ and $(1, 4, 2)$. $B = 330$ kN and acts through the point $(2, 6, 8)$ and $(4, 1, -2)$. $C = 85$ kN and acts through the point $(2, 6, 8)$ and $(2, -3, 3)$.

34. What is the x -component of the resultant force (kN)?

35. What is the y -component of the resultant force (kN)?

36. What is the z -component of the resultant force (kN)?

37. What is the resultant force (kN)?

Solution:

Let (2, 6, 8) be point O.

Determine the vectors created by forces A, B and C, respectively.

$$\overline{AO} = \langle 1 - 2 \quad 4 - 6 \quad 2 - 8 \rangle = \langle -1 \quad -2 \quad -6 \rangle$$

$$\overline{BO} = \langle 4 - 2 \quad 1 - 6 \quad -2 - 8 \rangle = \langle 2 \quad -5 \quad -10 \rangle$$

$$\overline{CO} = \langle 2 - 2 \quad -3 - 6 \quad 3 - 8 \rangle = \langle 0 \quad -9 \quad -5 \rangle$$

To get the x, y, and z components of the resultant force, multiply the forces to its respective unit vectors:

$$\vec{F} = 100 \frac{\langle -1 \quad -2 \quad -6 \rangle}{|\langle -1 \quad -2 \quad -6 \rangle|} + 330 \frac{\langle 2 \quad -5 \quad -10 \rangle}{|\langle 2 \quad -5 \quad -10 \rangle|} + 85 \frac{\langle 0 \quad -9 \quad -5 \rangle}{|\langle 0 \quad -9 \quad -5 \rangle|}$$

$$\vec{F} = \langle F_x \quad F_y \quad F_z \rangle = \langle 42.492 \quad -250.813 \quad -425.533 \rangle$$

To solve for the resultant of the force, get the absolute value of the vector answer:

$$\text{Abs}(\text{VctAns}) = 495.773 \text{ kN}$$

DIFFERENTIAL CALCULUS

1. Evaluate $\lim_{x \rightarrow 2} x^2 + 3x + 1$.

A. 11	C. 12
B. -11	D. 13

2. Evaluate $\lim_{x \rightarrow 0} \frac{1}{x}$.

A. undefined	C. 0
B. 1	D. no limit

3. Evaluate $\lim_{x \rightarrow 0} \frac{1}{x^2}$.

A. 0	C. 1
B. $+\infty$	D. $-\infty$

4. Evaluate $\lim_{x \rightarrow 9} \frac{x^2 - 81}{x - 9}$.

A. indeterminate	C. undefined
B. 0	D. 18

5. Evaluate $\lim_{x \rightarrow \infty} \frac{2x^4}{3x^4 + x^3 - x^2 - x - 1}$.

A. $+\infty$	C. 0
B. $2/3$	D. indeterminate

6. Evaluate $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$.

A. no limit	C. 0
B. indeterminate	D. $1/2$

7. Evaluate $\lim_{n \rightarrow \infty} \frac{\sin(\frac{2}{n})}{\sin(\frac{1}{n})}$.

A. 0	C. indeterminate
B. 2	D. no limit

8. Evaluate $\frac{dy}{dx}$, $y = x^2 - 5x$.

A. $2x$	C. $x - 5$
B. $2x - 5$	D. x

9. Evaluate $\frac{dy}{dx}$, $y = \frac{3}{x}$.

A. $\frac{3}{x^2}$	C. 3
B. $-\frac{3}{x^2}$	D. -3

10. If $f(x) = 3x^2 - 2x + 1$, find the value of x for which $f'(x) = 0$.

A. $1/3$	C. $-1/3$
B. $2/3$	D. 1

31. Find the angle of intersection of the curve:
 $y = 2x; x^5 + y^5 = 33$
 A. 63.43° C. 3.58°
 B. 67.01° D. 45°
32. Find the angle of intersection of the curve:
 $xy + y = 1; y^3 = (x + 1)^2$
 A. 45° C. 33.69°
 B. 11.31° D. 78.69°
33. Find the point of the ellipse $x^2 - 2xy + 4y^2 = 12$ at the right of y-axis where the abscissa x has its greatest value.
 A. (1, 4) C. (2, 1)
 B. (1, 2) D. (4, 1)
34. Find the points of inflection of $x^2 + y^{\frac{2}{3}} = 2$ at the first quadrant.
 A. (1,1) C. (1, -1)
 B. (-1, -1) D. (-1, 1)
35. Find the slope of the curve with parametric equation when $u = 2$:
 $x = 2u^3 - 3u^2; y = u^3 - 3u$
 A. 1 C. undefined
 B. $3/4$ D. $1/3$
36. Find the radius of curvature of the curve $y = x^4 - x^2$ at point (0, 0).
 A. $1/4$ C. $1/3$
 B. $3/4$ D. $1/2$
37. Find the radius of curvature of the curve $y = \frac{x}{x+1}$ at pt (0,0).
 A. 1.414 C. 1.5
 B. 0.354 D. 1.933
38. Find the radius of curvature of the curve $x^3 + y^3 = 4xy$ at point (2, 2).
 A. 1.414 C. 1.5
 B. 0.3554 D. 1.933
39. If the radius of the circle increases at the rate of 0.01 inch per second, find the rate of change of the area when the radius is 3 inches long.
 A. $0.188 \text{ in}^2/\text{s}$ C. $0.376 \text{ in}^2/\text{s}$
 B. $0.440 \text{ in}^2/\text{s}$ D. $0.335 \text{ in}^2/\text{s}$
40. A point moves on the parabola $y = x^2$ so that its abscissa increases at the rate of 3 feet per second. At what rate is the ordinate increasing when $x = 2$?
 A. 10 ft/s C. 15 ft/s
 B. 12 ft/s D. 8 ft/s

41. One leg of a right triangle is always 6 feet long and the other leg is increasing at the rate of 2 feet per second. Find the rate of change of the hypotenuse when it is 10 feet long?
- A. 1.6 ft/s
 B. 1.2 ft/s
 C. 3.2 ft/s
 D. 2.5 ft/s
42. A light hangs 15 feet directly above a straight walk on which a man 6 feet tall is walking. How fast is the end of the man's shadow traveling when he is walking away from the light at the rate of 3 miles per hour?
- A. 2.5 mph
 B. 2 mph
 C. 2.75 mph
 D. 1.5 mph
43. The base diameter and altitude of a right circular cone are observed at a certain instant to be 10 and 20 inches, respectively. If the lateral area is constant and the base diameter is increasing at the rate of 1 inch per minute, find the rate at which the altitude is decreasing.
- A. 2.25 in/min
 B. 7.5 in/min
 C. 5 in/min
 D. 4.32 in/min
44. A clock has hands 1 and $1\frac{3}{5}$ inches long respectively. At what rate are the ends of the hands approaching each other when the time is 2 o'clock?
- A. 5.70 in/s
 B. 3.29 in/s
 C. -5.70 in/s
 D. -3.29 in/s
45. A ladder 20 m long stands on a horizontal floor and leans against a vertical wall. If the top of the ladder slides at the rate of $\frac{1}{2}$ m/sec, find the rate at which the angle between the ladder and the wall is changing when the foot of the ladder is 12 m from the wall.
- A. 0.1 rad/s
 B. 0.3 rad/s
 C. 0.2 rad/s
 D. 0.4 rad/s
46. A conical glass whose radius is 5 m and altitude 12 m is being filled at the rate of 10 cu. m/sec. How fast is the surface rising when the liquid is 6 m deep?
- A. 3.18 m/s
 B. 0.51 m/s
 C. 2.12 m/s
 D. 0.13 m/s
47. $r = \frac{2}{2+\cos\theta}$, find the slope of the curve at $\theta = \frac{\pi}{3}$.
- A. 1.155
 B. 0.866
 C. -1.155
 D. -0.866
48. Find the angle between the radius vector and tangent line at the point indicated $r = a \sec 2\theta$, $\theta = \frac{\pi}{8}$.
- A. 26.57°
 B. 31.24°
 C. 27.66°
 D. 19.55°

DIFFERENTIAL CALCULUS SOLUTIONS

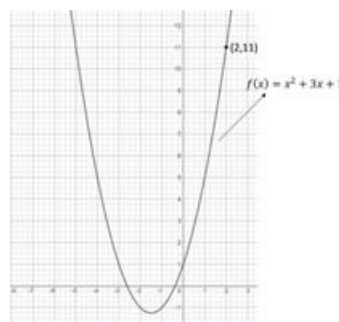
1. Evaluate $\lim_{x \rightarrow 2} x^2 + 3x + 1$.

Solution:

$$f(x) = x^2 + 3x + 1$$

$$f(2) = 2^2 + 3(2) + 1$$

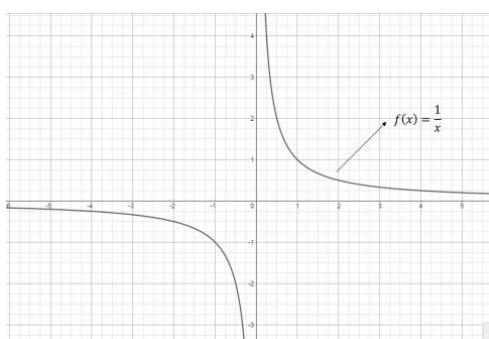
$$f(2) = \mathbf{11}$$



2. Evaluate $\lim_{x \rightarrow 0} \frac{1}{x}$.

Solution:

Assigning a very small number close to zero, could be -0.00001 or +0.00001. It is very important to check the values near the limit for both left and right of the limit.



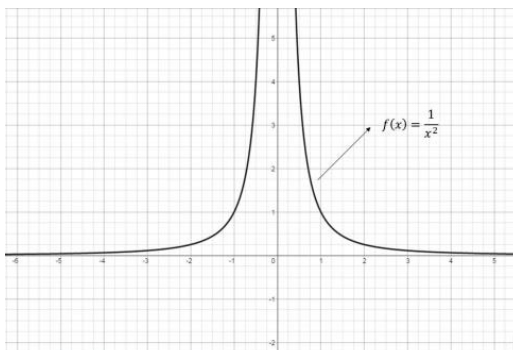
$$\lim_{x \rightarrow 0^-} \frac{1}{x} = \frac{1}{-0.00001} = -100000 \approx -\infty$$

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \frac{1}{+0.00001} = +100000 \approx +\infty$$

It shows that when x came from left of zero the function approaches negative infinity; and if x came from right of zero the function approached positive infinity; the function approaches different values. Therefore, the **limit does not exist**.

3. Evaluate $\lim_{x \rightarrow 0} \frac{1}{x^2}$.

Solution:



If x came from left of zero:

$$\frac{1}{x^2} \approx +\infty$$

If x came from right of zero:

$$\frac{1}{x^2} \approx +\infty$$

Hence,

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty$$

4. Evaluate $\lim_{x \rightarrow 9} \frac{x^2 - 81}{x - 9}$.

Solution:

$$\lim_{x \rightarrow 9} \frac{x^2 - 81}{x - 9} = \frac{9^2 - 81}{9 - 9} = \frac{0}{0} \rightarrow \text{indeterminate}$$

But:

$$\lim_{x \rightarrow 9} \frac{(x - 9)(x + 9)}{x - 9} = \lim_{x \rightarrow 9} x + 9 = \mathbf{18}$$

Alternative solution:

Try $x = 8.999$

Go to MODE 1 and type:

$$\frac{x^2 - 81}{x - 9}$$

Press CALC:

$$x? x = 8.999$$

The answer will be 17.999.

Try $x = 9.001$

Again, press CALC:

$$x? x = 9.001$$

The answer will be 18.001.

Since both limits approaches 18, therefore,

$$\lim_{x \rightarrow 9} \frac{x^2 - 81}{x - 9} = \mathbf{18}$$

5. Evaluate $\lim_{x \rightarrow \infty} \frac{2x^4}{3x^4 + x^3 - x^2 - x - 1}$.

Solution:

$$\lim_{x \rightarrow \infty} \frac{2x^4}{3x^4 + x^3 - x^2 - x - 1}$$

Dividing the numerator and denominator by the term with highest exponent:

$$\lim_{x \rightarrow \infty} \frac{2x^4}{3x^4 + x^3 - x^2 - x - 1} = \lim_{x \rightarrow \infty} \frac{(2x^4) \left(\frac{1}{x^4}\right)}{(3x^4 + x^3 - x^2 - x - 1) \left(\frac{1}{x^4}\right)}$$

$$\lim_{x \rightarrow \infty} \frac{2x^4}{3x^4 + x^3 - x^2 - x - 1} = \lim_{x \rightarrow \infty} \frac{2}{3 + \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{x^4}}$$

$$\lim_{x \rightarrow \infty} \frac{2x^4}{3x^4 + x^3 - x^2 - x - 1} = \frac{2}{3 + 0 - 0 - 0 - 0}$$

$$\lim_{x \rightarrow \infty} \frac{2x^4}{3x^4 + x^3 - x^2 - x - 1} = \frac{2}{3}$$

Alternative solution:

Go to MODE 1 and type:

$$\frac{2x^4}{3x^4 + x^3 - x^2 - x - 1}$$

Then press CALC:

$$x? x = 1000000 \rightarrow x \text{ is a very large value}$$

The answer will be $0.6666 \approx \mathbf{2/3}$.

6. Evaluate $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$.

Solution:

Note: In performing calculation involving trigonometric functions, the angle should be in radian mode.

Go to MODE 1 and type:

$$\frac{1 - \cos x}{x^2}$$

CALC:

$$\begin{aligned} x? x &= 0.001 \\ &= 0.4999 \approx 0.5 \end{aligned}$$

CALC:

$$\begin{aligned} x? x &= -0.001 \\ &= 0.4999 \approx 0.5 \end{aligned}$$

Same value, hence:

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \mathbf{0.5}$$

7. Evaluate $\lim_{n \rightarrow \infty} \frac{\sin(\frac{2}{n})}{\sin(\frac{1}{n})}$.

Solution:

Go to MODE 1 and type:

$$\frac{\sin(\frac{2}{x})}{\sin(\frac{1}{x})}$$

CALC:

$$\begin{aligned} x? x &= 1000000 \\ &= 1.999 \approx \mathbf{2} \end{aligned}$$

8. Evaluate $\frac{dy}{dx}$, $y = x^2 - 5x$.

Solution:

$$\begin{aligned} y &= x^2 - 5x \\ y' &= \mathbf{2x - 5} \end{aligned}$$

9. Evaluate $\frac{dy}{dx}$, $y = \frac{3}{x}$.

Solution:

$$\begin{aligned} y &= \frac{3}{x} \\ y &= 3x^{-1} \\ y' &= -3x^{-2} \\ y' &= \mathbf{-\frac{3}{x^2}} \end{aligned}$$

10. If $f(x) = 3x^2 - 2x + 1$, find the value of x for which $f'(x) = 0$.

Solution:

$$\begin{aligned} f(x) &= 3x^2 - 2x + 1 \\ f'(x) &= 6x - 2 \\ f'(x) &= 0 \\ 6x - 2 &= 0; x = \mathbf{\frac{1}{3}} \end{aligned}$$

11. Find the rate of change of area of the circle with respect to its radius when the radius is 3 feet.

Solution:

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$\frac{dA}{dr} = 2\pi(3)$$

$$\frac{dA}{dr} = \mathbf{6\pi \text{ ft}^2/\text{ft}}$$

12. Evaluate $\frac{dy}{dx}$, $y = (x - 1)(2x^2 + 3)$.

Solution:

$$y = (x - 1)(2x^2 + 3)$$

$$y' = (x - 1) \frac{d}{dx} (2x^2 + 3) + (2x^2 + 3) \frac{d}{dx} (x - 1)$$

$$y' = (x - 1)(4x) + (2x^2 + 3)$$

$$y' = 4x^2 - 4x + 2x^2 + 3$$

$$y' = \mathbf{6x^2 - 4x + 3}$$

13. Evaluate $\frac{dy}{dx}$, $y = 2x^{\frac{1}{2}} + 3x^{\frac{2}{3}}$.

Solution:

$$y = 2x^{\frac{1}{2}} + 3x^{\frac{2}{3}}$$

$$y' = 2 \cdot \frac{1}{2} x^{-\frac{1}{2}} + 3 \cdot \frac{2}{3} x^{-\frac{1}{3}}$$

$$y' = x^{-\frac{1}{2}} + 2x^{-\frac{1}{3}}$$

14. Evaluate $\frac{dy}{dx}$, $y = \frac{(2-x)(2x+1)}{x}$.

Solution:

$$y = \frac{(2 - x)(2x + 1)}{x}$$

$$y = \frac{-2x^2 + 3x + 2}{x}$$

$$y = -2x + 3 + \frac{2}{x}$$

$$y' = -2 - \frac{2}{x^2}$$

$$y' = -2 \left(1 + \frac{1}{x^2} \right)$$

15. Find $\frac{dy}{dx}$, $y = \frac{(1+\sqrt{x})(2+\sqrt{x})}{x}$, at $x = 1$.

Solution:

Use MODE 1 and type:

$$\frac{d}{dx} \left(\frac{(1 + \sqrt{x})(2 + \sqrt{x})}{x} \right)_{x=1} = \mathbf{-3.5}$$

16. Find the area of the circle when the rate of change of the area with respect to a diameter is 4π square feet per foot.

Solution:

$$A = \frac{\pi}{4} D^2$$

$$\frac{dA}{dD} = \frac{\pi}{4} \cdot 2D$$

$$\frac{dA}{dD} = \frac{\pi}{2} D$$

$$\frac{dA}{dD} = 4\pi$$

$$4\pi = \frac{\pi}{2} D$$

$$D = 8 \text{ ft}$$

Hence,

$$A = \frac{\pi}{4} D^2$$

$$A = \frac{\pi}{4} (8)^2$$

$$\mathbf{A = 16\pi \text{ square feet}}$$

17. Find the slope of the curve $y = 8x - 3x^2$ at pt. (2,4).

Solution:

$$y = 8x - 3x^2 \text{ at } (2,4)$$

$$y' = 8 - 6x$$

$$y' = 8 - 6(2)$$

$$\mathbf{y' = -4}$$

18. At what point is 2 the slope of the curve $y = 4x + x^2$?

Solution:

$$y = 4x + x^2$$

$$y' = 4 + 2x$$

$$y' = 2$$

$$2 = 4 + 2x$$

$$x = -1$$

$$y = 4(-1) + (-1)^2$$

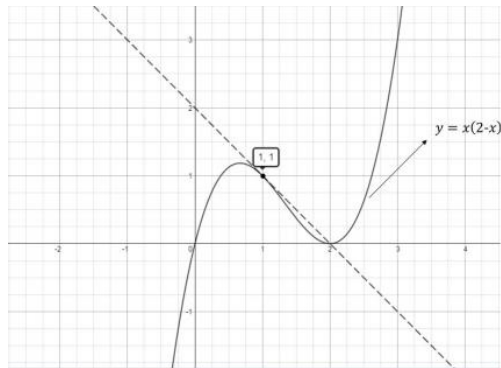
$$y = -3$$

Hence, the point is **(-1, -3)**.

19. Find the equation of the tangent line $y = x(2 - x)^2$ at pt (1,1).

Solution:

Solve first for the slope of the tangent line:



Go to MODE 1 and type:

$$\frac{d}{dx}(x(2-x)^2)_{x=1} = -1$$

Solving for the equation of the line: $m = -1$, point (1, 1):

$$y - 1 = (-1)(x - 1)$$

$$y - 1 = -x + 1$$

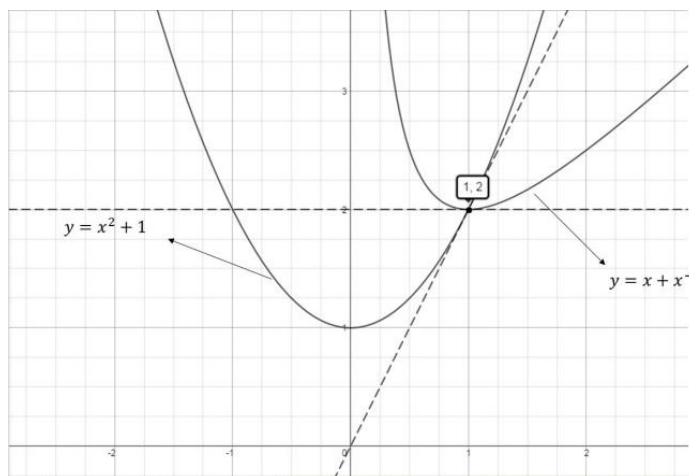
$$\mathbf{x + y - 2 = 0}$$

20. Find the angle of intersection between the curves

$$y = x^2 + 1$$

$$y = x + x^{-1}$$

Solution:



$$y = x^2 + 1 \rightarrow \text{equation 1}$$

$$y = x + x^{-1} \rightarrow \text{equation 2}$$

Solving for the point of intersection:

$$x^2 + 1 = x + x^{-1}$$

$$x^3 + x = x^2 + 1$$

$$x^3 - x^2 + x - 1 = 0$$

Solve using MODE 5-4:

$$[1 \quad -2 \quad 1 \quad -1]$$

$$x_1 = 1$$

$$x_2 = i$$

$$x_3 = -i$$

Solving for the value of y:

$$y = 1^2 + 1$$

$$y = 2$$

Therefore, the point of intersection is (1, 2).

Solving for the slope of tangent line of equation 1:

$$y' = 2x$$

$$y' = 2(1)$$

$$y' = 2$$

Solving for the slope of tangent line of equation 2:

$$y' = 1 - \frac{1}{x^2}$$

$$y' = 1 - \frac{1}{1^2}$$

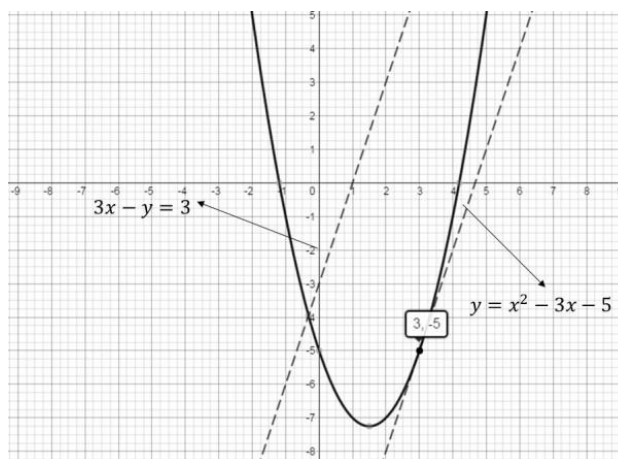
$$y' = 0$$

Referring to the figure:

$$\theta = \arctan(2 - 0) = 63.435^\circ$$

21. At what point of the parabola $y = x^2 - 3x - 5$ is the tangent line parallel to $3x - y = 3$?

Solution:



Solving for the slope of the line $3x - y = 2$:

$$y = 3x - 2$$

$$m = 3$$

Solving for the derivative:

$$y = x^2 - 3x - 5$$

$$y' = 2x - 3$$

Since the tangent line is parallel to $y = 3x - 3$, whose slope is 3, hence, $y' = 3$.

Solving for x:

$$3 = 2x - 3$$

$$x = 3$$

Now, solving for y:

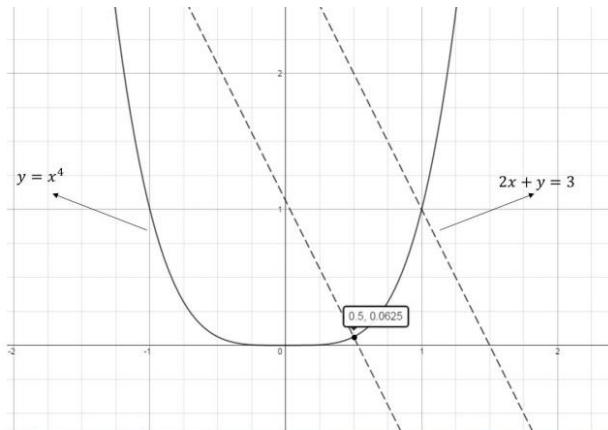
$$y = x^2 - 3x - 5$$

$$y = 3^2 - 3(3) - 5$$

$$y = -5$$

22. At what point of the curve $y = x^4$ is the normal line parallel to $2x + y = 3$?

Solution:



Solving for the slope of the line $2x + y = 3$:

$$y = -2x + 3$$

$$m = -2$$

Hence, the slope of normal line parallel to the line $2x + y = 3$ is -2 .

To solve for the slope of tangent line, take the negative reciprocal of the slope of the normal line:

$$m_T = \frac{1}{2}$$

$$y' = \frac{1}{2}$$

$$y' = 4x^3$$

$$\frac{1}{2} = 4x^3$$

$$x = \frac{1}{2}$$

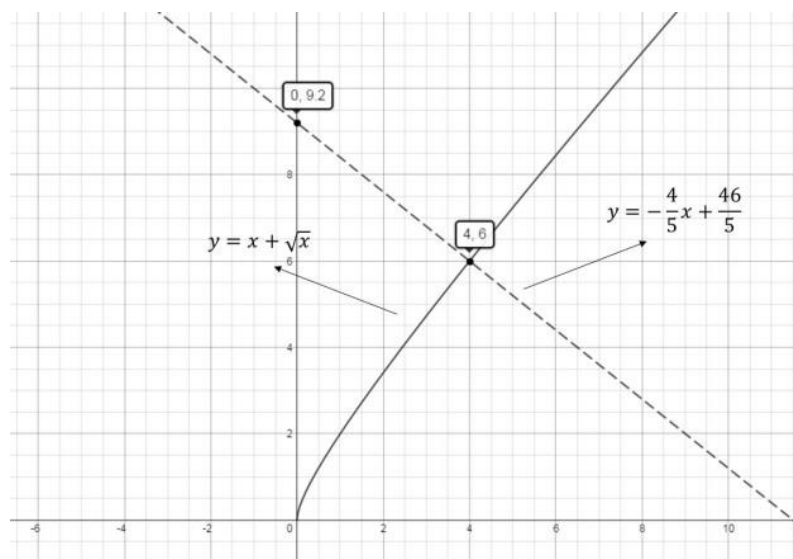
$$y = x^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

Therefore, the point is $\left(\frac{1}{2}, \frac{1}{16}\right)$.

23. Find the point where the normal to $y = x + \sqrt{x}$ at $(4,6)$ crosses the y-axis.

Solution:

Solve first for the slope of tangent line:



$$y = x + \sqrt{x} \text{ at } (4,6)$$

$$y' = 1 + \frac{1}{2\sqrt{x}}$$

$$y' = 1 + \frac{1}{2\sqrt{4}}$$

$$y' = \frac{5}{4}$$

Slope of normal line, $m_n = -\frac{4}{5}$.

Solve for the equation of the normal line: $m = -\frac{4}{5}$, pt (4,6)

$$y = mx + b$$

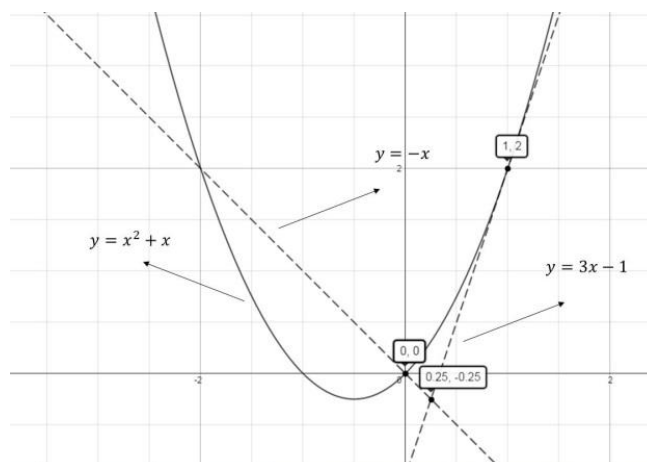
$$6 = -\frac{4}{5}(4) + b$$

$$b = \frac{46}{5}$$

The point where a line crosses the y-axis is its y-intercept, therefore the point is $(0, \frac{46}{5})$ or $(0, 9.2)$.

24. For the curve $y = x^2 + x$, at what point does the normal line at (0,0) intersect the tangent line at (1,2).

Solution:



Solving for the equation of a normal line through pt. (0, 0):

Solve firsts for the slope of the tangent line:

$$y = x^2 + x$$

$$y' = 2x + 1$$

$$y' = 2(0) + 1$$

$$y' = 1$$

Hence the slope of the normal line through (0, 0) is -1.

Solving now for the equation of a line through (0, 0):

$$m = -1, \text{ pt. } (0,0)$$

$$y = -x \rightarrow \text{equation 1}$$

Solving for the equation of the tangent line through (1, 2):

$$y = x^2 + x$$

$$y' = 2x + 1$$

$$y' = 2(1) + 1$$

$$y' = 3$$

Solving now for the equation of the line:

$$m = 3, \text{pt. } (1,2)$$

$$y - 2 = 3(x - 1)$$

$$y = 3x - 1 \rightarrow \text{equation 2}$$

Getting the intersection:

$$\begin{cases} y = -x \\ y = 3x - 1 \end{cases}$$

$$-x = 3x - 1$$

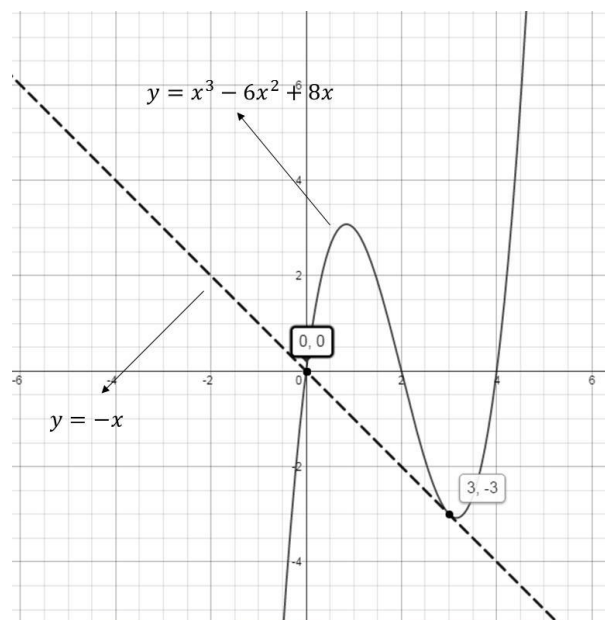
$$4x - 1 = 0$$

$$x = \frac{1}{4}; y = -\frac{1}{4}$$

Therefore, the point is $(\frac{1}{4}, \frac{1}{4})$.

25. The tangent to $y = x^3 - 6x^2 + 8x$ at $(3, -3)$ intersects the curve at another point. Find this point.

Solution:



Solve first for the equation of the tangent line at $(3, -3)$:

Solving the slope of tangent line, we have:

$$y = x^3 - 6x^2 + 8x$$

$$y' = 3x^2 - 12x + 8$$

$$y' = 3(3)^2 - 12(3) + 8$$

$$y' = -1$$

Solving now the equation of tangent line:

$$m = -1, \text{pt } (3, -3)$$

$$y + 3 = -(x - 3)$$

$$y = -x$$

Solving for the point of intersection:

$$y = -x$$

$$y = x^3 - 6x^2 + 8x$$

$$-x = x^3 - 6x^2 + 8x$$

$$x^3 - 6x^2 + 9x = 0$$

Go to MODE 5-4:

$$[1 \quad -6 \quad 9 \quad 0]$$

$$x_1 = 0, y_1 = 0; x_2 = 3, y_2 = -3$$

Therefore, the other point is $(0,0)$.

26. Find the point of inflection of the curve $y = 5 - 2x - 3x^2 - x^3$.

Solution:

For the point of inflection:

$$y'' = 0$$

$$y = 5 - 2x - 3x^2 - x^3$$

$$y' = -2 - 6x - 3x^2$$

$$y'' = -6 - 6x$$

$$0 = -6 - 6x$$

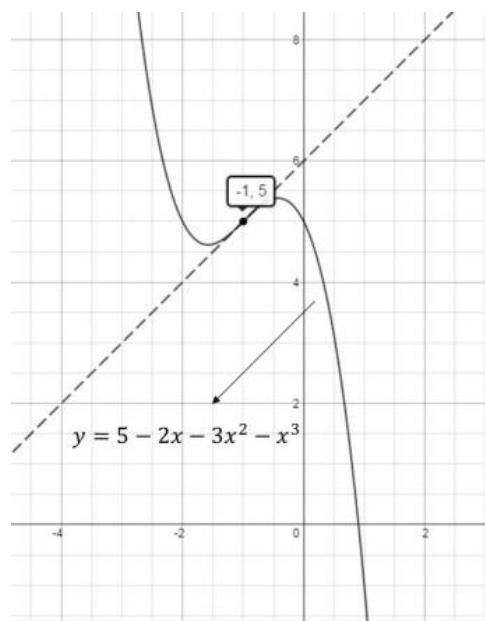
$$x = -1$$

Solving for y:

$$y = 5 - 2(-1) - 3(-1)^2 - (-1)^3$$

$$y = 5$$

Therefore, the point is **(-1, 5)**.



27. Find $\frac{dy}{dx}$, $x^2 - 4y^2 = 4$.

Solution:

$$\frac{d}{dx}[x^2 - 4y^2 = 4]$$

$$2x - 4yy' = 0$$

$$y' = \frac{2x}{8y}$$

$$y' = \frac{x}{4y}$$

28. Find $\frac{dy}{dx}$, $x^4 + y^3 = 2x^2y^2$.

Solution:

$$\frac{d}{dx}[x^4 + y^3 = 2x^2y^2]$$

$$4x^3 + 3y^2y' = 2[x^2(2yy') + y^2(2x)]$$

$$4x^3 + 3y^2y' = 4x^2yy' + 4xy^2$$

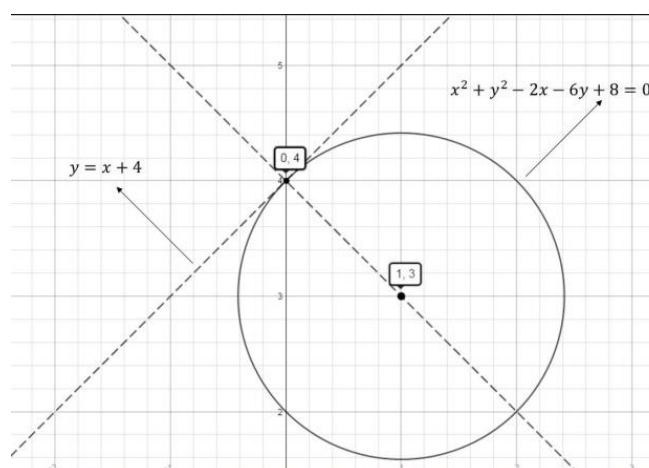
$$3y^2y' - 4x^2yy' = 4xy^2 - 4x^3$$

$$y'(3y^2 - 4x^2y) = 4xy^2 - 4x^3$$

$$y' = \frac{4xy^2 - 4x^3}{3y^2 - 4x^2y}$$

29. Find the equation of the of the tangent line to the curve $x^2 + y^2 - 2x - 6y + 8 = 0$ at (0,4).

Solution:



$$\frac{d}{dx} [x^2 + y^2 - 2x - 6y + 8 = 0] \text{ at } (0,4)$$

$$2x + 2yy' - 2 - 6y' = 0$$

$$2(0) + 2(4)y' - 2 - 6y' = 0$$

$$2y' = 2$$

$$y' = 1$$

Solving for the equation of tangent line:

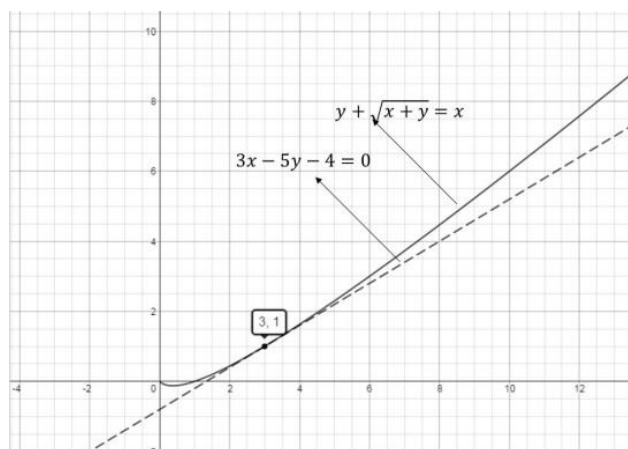
$$m = 1, \text{ pt. } (0,4)$$

$$y - 4 = x$$

$$y = x + 4$$

30. Find the equation of the tangent line to the curve $y + \sqrt{x+y} = x$ at pt. (3,1).

Solution:



$$\frac{d}{dx} [y + \sqrt{x+y} = x] \text{ at pt. } (3,1)$$

$$y' + \frac{1}{2}(x+y)^{-\frac{1}{2}}(1+y') = 1$$

$$y' + \frac{1}{2}(3+1)^{-\frac{1}{2}}(1+y') = 1$$

$$y' = \frac{3}{5}$$

Solving for the equation of the line:

$$m = \frac{3}{5}, \text{ pt. } (3,1)$$

$$y - 1 = \frac{3}{5}(x - 3)$$

$$5y - 5 = 3x - 9$$

$$3x - 5y - 4 = 0$$

31. Find the angle of intersection of the curve:

$$y = 2x$$

$$x^5 + y^5 = 33$$

Solution:

Solving for the point of intersection:

$$x^5 + (2x)^5 = 33$$

$$x^5 + 32x^5 = 33$$

$$33x^5 = 33$$

$$x^5 = 1$$

$$x = 1; y = 2$$

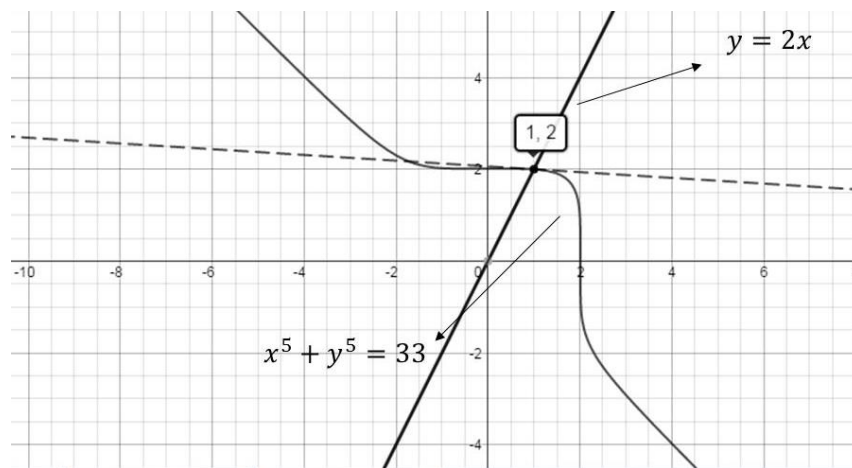
Now, solving for the slopes:

$$y = 2x$$

$$y'_1 = 2$$

$$\theta_1 = \tan^{-1}(2)$$

$$\theta_1 = 63.43^\circ$$



$$x^5 + y^5 = 33$$

$$5x^4 + 5y^4y' = 0$$

$$y' = -\frac{x^4}{y^4}$$

$$y' = -\frac{1^4}{2^4}$$

$$y'_2 = -\frac{1}{16}$$

$$\theta_2 = \tan^{-1}\left(\frac{1}{16}\right)$$

$$\theta_2 = -3.58^\circ$$

Therefore, the angle of intersection is:

$$\theta = |\theta_1 - \theta_2|$$

$$\theta = |63.43^\circ - (-3.58^\circ)|$$

$$\theta = 67.01^\circ$$

Alternative solution in solving for the angle of intersection:

$$\tan \theta = \frac{y'_1 - y'_2}{1 + y'_1y'_2}$$

$$\tan \theta = \frac{2 - \left(-\frac{1}{16}\right)}{1 + 2\left(-\frac{1}{16}\right)}$$

$$\tan \theta = \frac{33}{14}$$

$$\theta = 67.01^\circ$$

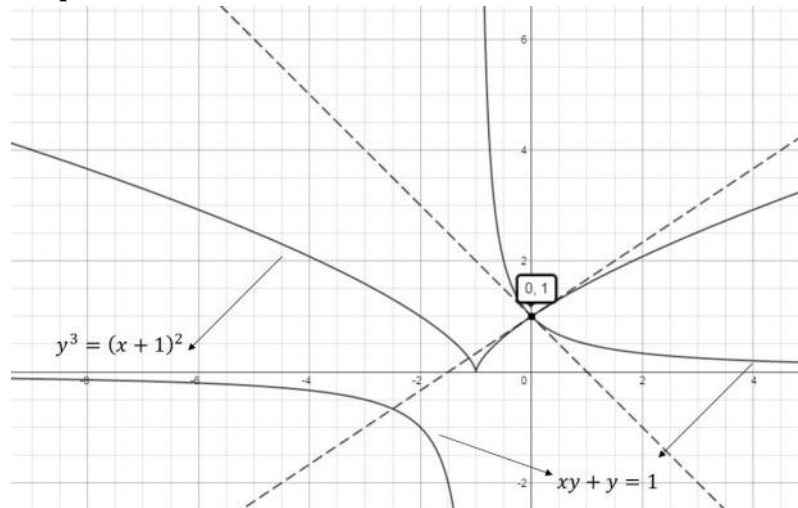
32. Find the angle of intersection of the curve:

$$xy + y = 1$$

$$y^3 = (x + 1)^2$$

Solution:

Solving for the point of intersection:



$$xy + y = 1$$

$$y(x + 1) = 1$$

$$x + 1 = \frac{1}{y}$$

Substituting it to the other equation:

$$y^3 = \left(\frac{1}{y}\right)^2$$

$$y^3 = \frac{1}{y^2}$$

$$y^5 = 1$$

$$y = 1; x = 0$$

Now, solving for the slopes:

$$xy + y = 1$$

$$xy' + y + y' = 0$$

$$(0)y' + 1 + y' = 0$$

$$y'_1 = -1$$

$$\theta_1 = -45^\circ$$

$$y^3 = (x + 1)^2$$

$$3y^2y' = 2(x + 1)$$

$$3(1)^2y' = 2(0 + 1)$$

$$3y' = 2$$

$$y'_2 = \frac{2}{3}$$

$$\theta_2 = 33.69^\circ$$

Therefore, the angle of intersection is:

$$\theta = |\theta_1 - \theta_2|$$

$$\theta = |-45^\circ - 33.69^\circ|$$

$$\theta = 78.69^\circ$$

Alternative solution in solving for the angle of intersection:

$$\tan \theta = \frac{y'_1 - y'_2}{1 + y'_1 y'_2}$$

$$\tan \theta = \frac{-1 - \frac{2}{3}}{1 + (-1)\left(\frac{2}{3}\right)}$$

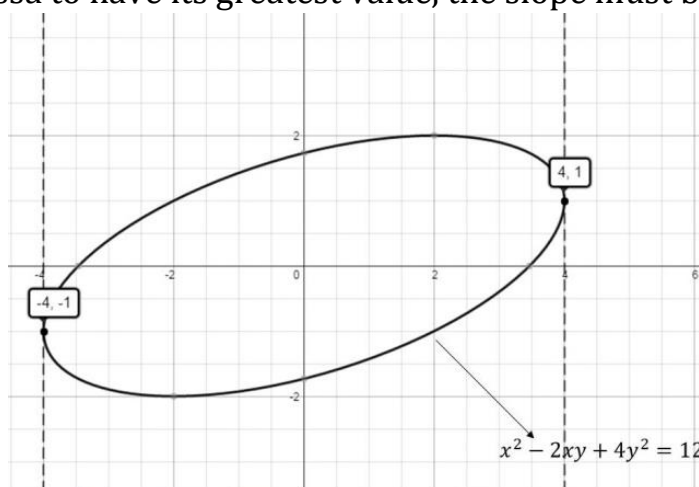
$$\tan \theta = -5$$

$$\theta = -78.69^\circ \text{ (just take the positive value, } 78.69^\circ)$$

33. Find the points of the ellipse $x^2 - 2xy + 4y^2 = 12$ at the right of y-axis where the abscissa x has its greatest value.

Solution:

For the abscissa to have its greatest value, the slope must be undefined:



Therefore:

$$\frac{dx}{dy} = 0$$

$$x^2 - 2xy + 4y^2 = 12$$

$$2x \frac{dx}{dy} - 2\left(\frac{dx}{dy}y + x\right) + 8y = 0$$

$$2x(0) - 2[(0)y + x] + 8y = 0$$

$$-2x + 8y = 0$$

$$x = 4y$$

Substituting it to the equation of ellipse:

$$(4y)^2 - 2(4y)(y) + 4y^2 = 12$$

$$16y^2 - 8y^2 + 4y^2 = 12$$

$$12y^2 = 12$$

$$y^2 = 1$$

$$y = \pm 1$$

$$x = 4(\pm 1)$$

$$x = \pm 4$$

Therefore, the point is **(4,1)**.

34. Find the point of inflection of $x^2 + y^{\frac{2}{3}} = 2$ at the first quadrant.

Solution:

$$2x + \frac{2}{3}y^{-\frac{1}{3}}y' = 0$$

$$2xy^{\frac{1}{3}} + \frac{2}{3}y' = 0$$

$$y' = -3xy^{\frac{1}{3}}$$

$$y'' = -3 \left[x \left(\frac{1}{3}y^{-\frac{2}{3}}y' \right) + y^{\frac{1}{3}} \right]$$

$$y'' = -3 \left\{ x \left[\frac{1}{3}y^{-\frac{2}{3}} \left(-3xy^{\frac{1}{3}} \right) \right] + y^{\frac{1}{3}} \right\}$$

$$y'' = -3 \left[-x^2y^{-\frac{1}{3}} + y^{\frac{1}{3}} \right]$$

For the point of inflection:

$$y'' = 0$$

Therefore:

$$0 = x^2y^{-\frac{1}{3}} - y^{\frac{1}{3}}$$

$$x^2y^{-\frac{1}{3}} = y^{\frac{1}{3}}$$

$$x^2 = y^{\frac{2}{3}}$$

But:

$$y^{\frac{2}{3}} = 2 - x^2$$

Substituting:

$$x^2 = 2 - x^2$$

$$2x^2 = 2$$

$$x^2 = 1$$

$$x = \pm 1$$

Solving for y:

$$y = \pm 1$$

Therefore, the point of inflection is **(1,1)**.

35. Find the slope of the curve with parametric equation when $u = 2$:

$$x = 2u^3 - 3u^2$$

$$y = u^3 - 3u$$

Solution:

$$\frac{dx}{du} = 6u^2 - 6u$$

$$\frac{dx}{du} = 6(2)^2 - 6(2)$$

$$\frac{dx}{du} = 12$$

$$\frac{dy}{du} = 3u^2 - 3$$

$$\frac{dy}{du} = 3(2)^2 - 3$$

$$\frac{dy}{du} = 9$$

Therefore:

$$\frac{dy}{dx} = \frac{\frac{dy}{du}}{\frac{dx}{du}} = \frac{9}{12} = \frac{3}{4}$$

Alternative solution:

$$\begin{aligned} \frac{d}{dx}(2X^3 - 3X^2)_{X=2} &= 12 \rightarrow \frac{dx}{du} \\ \frac{d}{dx}(X^3 - 3X)_{X=2} &= 9 \rightarrow \frac{dy}{du} \end{aligned}$$

Therefore:

$$\frac{dy}{dx} = \frac{9}{12} = \frac{3}{4} \text{ (ok!)}$$

36. Find the radius of curvature of the curve $y = x^4 - x^2$ at pt (0,0).

Solution:

$$\rho = \frac{[1 + (y')^2]^{\frac{3}{2}}}{|y''|}$$

Solving y' and y'' :

$$\begin{aligned} y' &= 4x^3 - 2x \\ y' &= 4(0)^3 - 2(0) \\ y' &= 0 \\ y'' &= 12x^2 - 2 \\ y'' &= 12(0)^2 - 2 \\ y'' &= -2 \end{aligned}$$

Therefore:

$$\begin{aligned} \rho &= \frac{[1 + 0^2]^{\frac{3}{2}}}{|-2|} \\ \rho &= \frac{1}{2} \end{aligned}$$

37. Find the radius of curvature of the curve $y = \frac{x}{x+1}$ at pt (0,0).

Solution:

$$\begin{aligned} y' &= \frac{(x+1)(1) - (x)(1)}{(x+1)^2} \\ y' &= \frac{x+1-x}{(x+1)^2} \\ y' &= \frac{1}{(x+1)^2} \\ y' &= \frac{1}{(0+1)^2} \\ y' &= 1 \end{aligned}$$

$$\begin{aligned} y'' &= \frac{-2}{(x+1)^3} \\ y'' &= \frac{-2}{(0+1)^3} \\ y'' &= -2 \end{aligned}$$

Therefore:

$$\rho = \frac{[1 + (1)^2]^{\frac{3}{2}}}{|-2|}$$

$$\rho = \sqrt{2}$$

38. Find the radius of curvature of the curve $x^3 + y^3 = 4xy$ at pt (2,2).

Solution:

$$\frac{d}{dx}[x^3 + y^3 = 4xy] \text{ at } (2,2)$$

$$3x^2 + 3y^2y' = 4(xy' + y)$$

$$3x^2 + 3y^2y' = 4xy' + 4y$$

$$3(2)^2 + 3(2)^2y' = 4(2)y' + 4(2)$$

$$y' = -1$$

$$6x + 3[y^2y'' + y'(2yy')] = 4(xy'' + y') + 4y'$$

$$6x + 3y^2y'' + 6y(y')^2 = 4xy'' + 8y'$$

$$6(2) + 3(2)^2y'' + 6(2)(-1)^2 = 4(2)y'' + 8(-1)$$

$$y'' = -8$$

Therefore:

$$\rho = \frac{[1 + (-1)^2]^{\frac{3}{2}}}{|-8|}$$

$$\rho = \frac{\sqrt{2}}{4}$$

39. If the radius of the circle increases at the rate of 0.01 inch per second, find the rate of change of the area when the radius is 3 inches long.

Solution:

Given:

$$\frac{dr}{dt} = 0.01 \frac{\text{in}}{\text{s}}$$

Required:

$$\frac{dA}{dt} = ? \text{ when } r = 3 \text{ in}$$

From the area of circle:

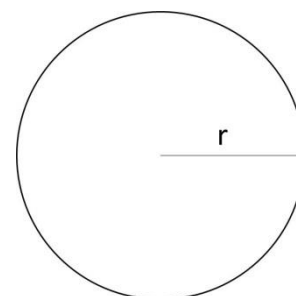
$$A = \pi r^2$$

Differentiating the whole equation with respect to time, we have:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi(3)(0.01 \text{ in/s})$$

$$\frac{dA}{dt} = \frac{3\pi}{50} \approx \mathbf{0.188 \text{ in}^2/\text{s}}$$



40. A point moves on the parabola $y = x^2$ so that its abscissa increases at the rate of 3 feet per second. At what rate is the ordinate increasing when $x = 2$?

Solution:

Given:

$$\frac{dx}{dt} = 3 \text{ ft/s}$$

Required:

$$\frac{dy}{dt} = ? \text{ when } x = 2$$

Differentiating the whole equation with respect to time:

$$\frac{dy}{dt} = 2x \frac{dx}{dt}$$

$$\frac{dy}{dt} = 2(2)(3)$$

$$\frac{dy}{dt} = 12 \text{ ft/s}$$

41. One leg a right triangle is always 6 feet long and the other leg is increasing at the rate of 2 feet per second. Find the rate of change of the hypotenuse when it is 10 feet long?

Solution:

Referring to the figure:

By Pythagorean Theorem:

$$x^2 + 6^2 = L^2$$

When $L = 10$ ft:

$$x^2 = 10^2 - 6^2$$

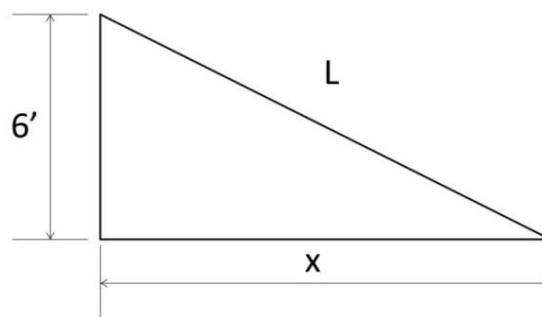
$$x = 8 \text{ ft}$$

Now, differentiating the whole equation with respect to time:

$$2x \frac{dx}{dt} = 2L \frac{dL}{dt}$$

$$2(8)(2 \text{ ft/s}) = 2(10) \frac{dL}{dt}$$

$$\frac{dL}{dt} = 1.6 \text{ ft/s}$$



42. A light hangs 15 feet directly above a straight walk on which a man 6 feet tall is walking. How fast is the end of the man's shadow traveling when he is walking away from the light at the rate of 3 miles per hour?

Solution:

Referring to the figure:

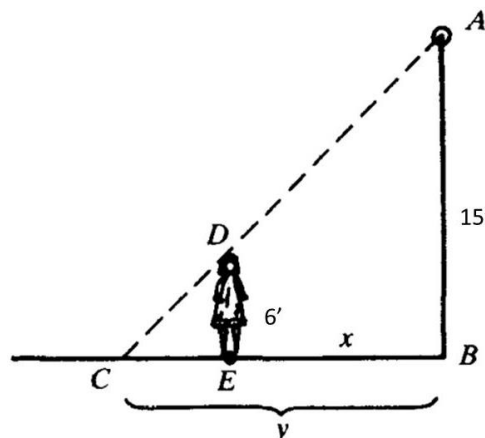
$$\frac{dx}{dt} = 3 \text{ mph}$$

$$\frac{dy}{dt} = ? \text{ when } \frac{dx}{dt} = 3 \text{ mph}$$

By similar triangles:

$$\frac{y-x}{9} = \frac{y}{15}$$

$$x = \frac{3}{2}y$$



Differentiating the whole equation with respect to time:

$$\frac{dx}{dt} = \frac{3 dy}{2 dt}$$

$$3 \frac{mi}{hr} = \frac{3 dy}{2 dt}$$

$$\frac{dy}{dt} = \mathbf{2 \text{ mph}}$$

43. The base diameter and altitude of a right circular cone are observed at a certain instant to be 10 and 20 inches, respectively. If the lateral area is constant and the base diameter is increasing at the rate of 1 inch per minute, find the rate at which the altitude is decreasing.

Solution:

From the general formula of lateral surface area:

$$A = \frac{1}{2}(P_1 + P_2)L$$

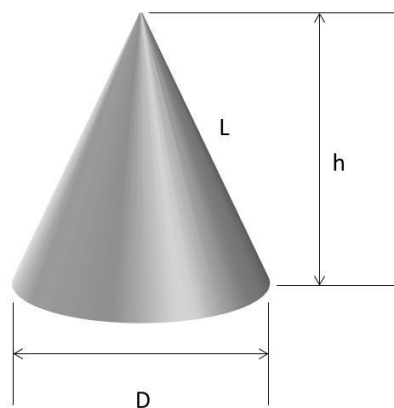
$$P_1 = \pi D$$

$$P_2 = 0$$

By Pythagorean Theorem:

$$L^2 = h^2 + \left(\frac{D}{2}\right)^2$$

$$L = \sqrt{h^2 + \left(\frac{D}{2}\right)^2}$$



Substituting:

$$A = \frac{1}{2} \pi D \sqrt{h^2 + \left(\frac{D}{2}\right)^2}$$

Differentiating the whole equation with respect to time:

$$\frac{dA}{dt} = \frac{\pi}{2} \left[D \cdot \frac{2h \frac{dh}{dt} + \frac{1}{4} \cdot 2D \frac{dD}{dt}}{2\sqrt{h^2 + \left(\frac{D}{2}\right)^2}} + \sqrt{h^2 + \left(\frac{D}{2}\right)^2} \cdot \frac{dD}{dt} \right]$$

Since the lateral area is constant:

$$\frac{dA}{dt} = 0$$

Therefore:

$$\frac{\pi}{2} \left[D \cdot \frac{2h \frac{dh}{dt} + \frac{1}{4} \cdot 2D \frac{dD}{dt}}{2\sqrt{h^2 + \left(\frac{D}{2}\right)^2}} + \sqrt{h^2 + \left(\frac{D}{2}\right)^2} \cdot \frac{dD}{dt} \right] = 0$$

$$D \cdot \frac{h \frac{dh}{dt} + \frac{1}{4} \cdot D \frac{dD}{dt}}{\sqrt{h^2 + \left(\frac{D}{2}\right)^2}} + \sqrt{h^2 + \left(\frac{D}{2}\right)^2} \cdot \frac{dD}{dt} = 0$$

Now, substitute the following:

$$D = 10 \text{ in}, h = 20 \text{ in}, \frac{dD}{dt} = 1 \text{ in/minute}$$

We have:

$$10 \cdot \frac{20 \frac{dh}{dt} + \frac{1}{4} \cdot (10)(1)}{\sqrt{20^2 + \left(\frac{10}{2}\right)^2}} + \sqrt{20^2 + \left(\frac{10}{2}\right)^2} \cdot (1) = 0$$

$$10 \left(20 \frac{dh}{dt} + 2.5 \right) + (20^2 + 5^2) = 0$$

$$10 \left(20 \frac{dh}{dt} + 2.5 \right) = -425$$

$$20 \frac{dh}{dt} + 2.5 = -42.5$$

$$20 \frac{dh}{dt} = -45$$

$$\frac{dh}{dt} = \mathbf{-2.25 \text{ in/minute}}$$

The negative sign shows that the height is decreasing at a rate of 2.25 inches per minute.

44. A clock has hands 1 and $1\frac{3}{5}$ inches long respectively. At what rate are the ends of the hands approaching each other when the time is 2 o'clock?

Solution:

Referring to the figure:

By cosine law:

$$x^2 = 1^2 + 1.6^2 - 2(1)(1.6) \cos 60^\circ$$

$$x = 1.4 \text{ in}$$

Differentiating:

$$2x \frac{dx}{dt} = 3.2 \sin \theta \frac{d\theta}{dt}$$

Remember that:

$$\frac{d\theta}{dt} = \pm 5.5^\circ/\text{min} = \pm \frac{11\pi}{360} \text{ rad/minute}$$

As the minute hand approaches the hour hand $\frac{d\theta}{dt}$ takes the negative value, $\frac{d\theta}{dt}$ is positive as it leaves the hour hand.

Therefore:

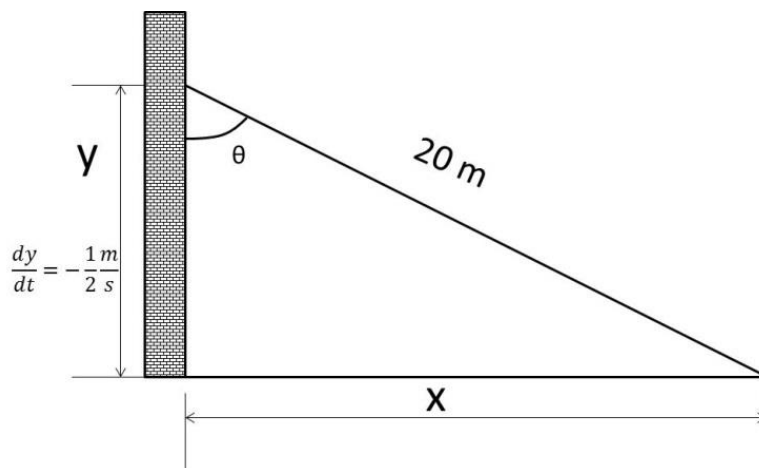
$$2(1.4) \frac{dx}{dt} = 3.2 \sin 60^\circ \left(-\frac{11\pi}{360} \text{ rad/min} \right)$$

$$\frac{dx}{dt} = -0.095 \text{ in/min}$$

$$\frac{dx}{dt} = \mathbf{-5.70 \text{ in/s}}$$

45. A ladder 20 m long stands on a horizontal floor and leans against a vertical wall. If the top of the ladder slides at the rate of $\frac{1}{2}$ m/sec, find the rate at which the angle between the ladder and the wall is changing when the foot of the ladder is 12 m from the wall.

Solution:



Required:

$$\frac{d\theta}{dt} = ? \text{ when } x = 12 \text{ m}$$

Referring to the figure:

By Pythagorean Theorem:

$$\begin{aligned} x^2 + y^2 &= 20^2 \\ y^2 &= 20^2 - 12^2 \\ y^2 &= 256 \\ y &= 16 \text{ m} \end{aligned}$$

Using trigonometric functions:

$$\cos \theta = \frac{y}{20}$$

Differentiating the whole equation with respect to time:

$$\begin{aligned} 20 \cos \theta &= y \\ -20 \sin \theta \cdot \frac{d\theta}{dt} &= \frac{dy}{dt} \end{aligned}$$

Using trigonometric functions:

$$\begin{aligned} \sin \theta &= \frac{x}{20} \\ \sin \theta &= \frac{12}{20} \\ \sin \theta &= \frac{3}{5} \end{aligned}$$

Substituting:

$$\begin{aligned} -20 \left(\frac{3}{5}\right) \frac{d\theta}{dt} &= -1.2 \\ -12 \frac{d\theta}{dt} &= -1.2 \\ \frac{d\theta}{dt} &= 0.1 \text{ rad/s} \end{aligned}$$

46. A conical glass whose radius is 5 m and altitude 12 m is being filled at the rate of 10 cu. m/sec. How fast is the surface rising when the liquid is 6 m deep?

Solution:

Referring to the figure:

$$\frac{dV}{dt} = 10 \text{ m}^3/\text{s}$$

Required:

$$\frac{dh}{dt} = ? \text{ when } h = 6 \text{ m}$$

From the formula of volume of a cone:

$$V = \frac{\pi}{3} r^2 h$$

Solve first the relationship between the radius and the height:

By similar triangles:

$$\frac{r}{h} = \frac{5}{12}$$

$$r = \frac{5}{12} h$$

Substituting:

$$V = \frac{\pi}{3} \left(\frac{5}{12} h \right)^2 h$$

$$V = \frac{25\pi}{432} h^3$$

Differentiating the whole equation with respect to time:

$$\frac{dV}{dt} = \frac{25\pi}{432} \cdot 3h^2 \frac{dh}{dt}$$

Thus,

$$10 \frac{\text{m}^3}{\text{s}} = \frac{25\pi}{432} \cdot 3(6)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \mathbf{0.509 \text{ m/s}}$$

Alternative solution:

From the formula:

$$Q = Av$$

By letting:

$$Q = \frac{dV}{dt}; A = \pi r^2; v = \frac{dh}{dt}$$

When $h = 6 \text{ m}$:

$$r = \frac{5}{12} h$$

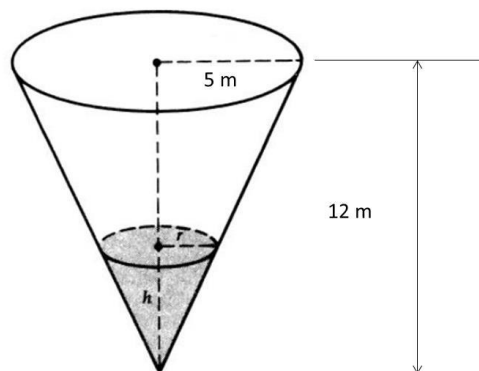
$$r = \frac{5}{12} (6)$$

$$r = 2.5 \text{ m}$$

Therefore:

$$10 \frac{\text{m}^3}{\text{s}} = \pi (2.5)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \mathbf{0.509 \text{ m/s (ok!)}$$



47. $r = \frac{2}{2 + \cos \theta}$, find the slope of the curve at $\theta = \frac{\pi}{3}$.

Solution:

Remember, in rectangular form:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

Differentiating both equations with respect to θ :

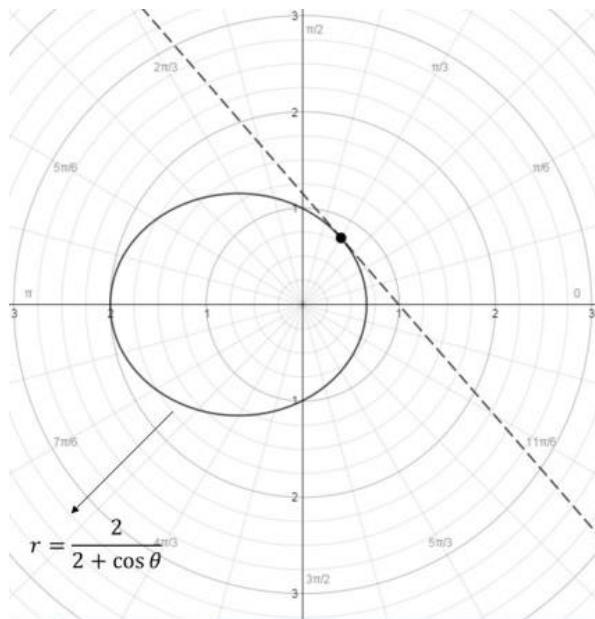
$$\frac{dx}{d\theta} = r(-\sin \theta) + \cos \theta \frac{dr}{d\theta}$$

$$\frac{dy}{d\theta} = r \cos \theta + \sin \theta \frac{dr}{d\theta}$$

Therefore, the slope is:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\sin \theta \frac{dr}{d\theta} + r \cos \theta}{\cos \theta \frac{dr}{d\theta} - r \sin \theta}$$

Thus:



$$\frac{dr}{d\theta} = 2(-1)(2 + \cos \theta)^{-2}(-\sin \theta)$$

$$\frac{dr}{d\theta} = \frac{2 \sin \theta}{(2 + \cos \theta)^2}$$

Substituting $\theta = \frac{\pi}{3}$ to r and $\frac{dr}{d\theta}$:

$$r = \frac{2}{2 + \cos \frac{\pi}{3}}$$

$$r = \frac{4}{5}$$

$$\frac{dr}{d\theta} = \frac{2 \sin \frac{\pi}{3}}{\left(2 + \cos \frac{\pi}{3}\right)^2}$$

$$\frac{dr}{d\theta} = \frac{4\sqrt{3}}{25}$$

Hence,

$$\frac{dy}{dx} = \frac{\left(\sin \frac{\pi}{3}\right)\left(\frac{4\sqrt{3}}{25}\right) + \left(\frac{4}{5}\right)\left(\cos \frac{\pi}{3}\right)}{\left(\cos \frac{\pi}{3}\right)\left(\frac{4\sqrt{3}}{25}\right) - \left(\frac{4}{5}\right)\left(\sin \frac{\pi}{3}\right)}$$

$$\frac{dy}{dx} = \frac{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{4\sqrt{3}}{25}\right) + \left(\frac{4}{5}\right)\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)\left(\frac{4\sqrt{3}}{25}\right) - \left(\frac{4}{5}\right)\left(\frac{\sqrt{3}}{2}\right)}$$

$$\frac{dy}{dx} = -\frac{2\sqrt{3}}{3} \approx -1.155$$

Alternative solution:

Be sure that your calculator is in radian mode:

$$\frac{d}{dx} \left(\frac{2}{2 + \cos X} \cos X \right)_{X=\frac{\pi}{3}} = -0.554 \rightarrow \frac{dx}{d\theta}$$

$$\frac{d}{dx} \left(\frac{2}{2 + \cos X} \sin X \right)_{X=\frac{\pi}{3}} = 0.64 \rightarrow \frac{dy}{d\theta}$$

Therefore:

$$\frac{dy}{dx} = \frac{0.64}{-0.554} = -1.155$$

48. Find the angle between the radius vector and tangent line at the point indicated $r = a \sec 2\theta$, $\theta = \frac{\pi}{8}$.

Solution:

The angle β between the tangent and the radius vector is:

$$\tan \beta = \frac{r}{dr/d\theta}$$

$$r = a \sec 2\theta$$

$$\frac{dr}{d\theta} = 2a \sec 2\theta \tan 2\theta$$

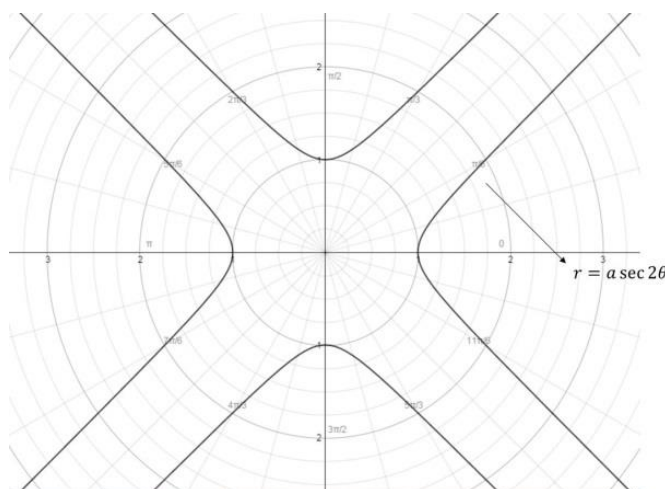
Therefore:

$$\tan \beta = \frac{a \sec 2\theta}{2a \sec 2\theta \tan 2\theta}$$

$$\tan \beta = \frac{1}{2 \tan 2\theta}$$

$$\tan \beta = \frac{1}{2 \tan \left[2 \left(\frac{\pi}{8} \right) \right]}$$

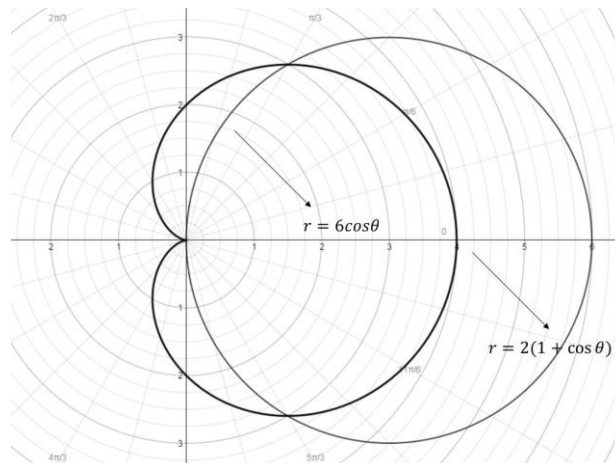
$$\beta = 26.565^\circ$$



49. Find the angle of intersection between the curve:

$$r = 6 \cos \theta$$

$$r = 2(1 + \cos \theta)$$



Solution:

Solving first for the point(s) of intersection:

$$6 \cos \theta = 2(1 + \cos \theta)$$

$$3 \cos \theta = 1 + \cos \theta$$

$$2 \cos \theta = 1$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}; \theta = \frac{5\pi}{3}$$

Referring to the figure:

Since both graphs are symmetrical with respect to the x-axis; also, the points of intersection are symmetrical with respect to x-axis, we can say that they form an equal angle of intersection:

$$r_1 = 6 \cos \theta$$

$$r_1 = 6 \cos \frac{\pi}{3}$$

$$r_1 = 3$$

$$\frac{dr_1}{d\theta} = -6 \sin \theta$$

$$\frac{dr_1}{d\theta} = -6 \sin \frac{\pi}{3}$$

$$\frac{dr_1}{d\theta} = -3\sqrt{3}$$

$$y'_1 = \frac{\sin \theta \frac{dr_1}{d\theta} + r_1 \cos \theta}{\cos \theta \frac{dr_1}{d\theta} - r_1 \sin \theta}$$

$$y'_1 = \frac{\left(\sin \frac{\pi}{3}\right) (-3\sqrt{3}) + 3 \cos \frac{\pi}{3}}{\left(\cos \frac{\pi}{3}\right) (-3\sqrt{3}) - 3 \sin \frac{\pi}{3}}$$

$$y'_1 = \frac{\sqrt{3}}{3}$$

$$\theta_1 = 30^\circ$$

$$r_2 = 2(1 + \cos \theta)$$

$$r_2 = 2 \left(1 + \cos \frac{\pi}{3} \right)$$

$$r_2 = 3$$

$$\frac{dr_2}{d\theta} = 2(-\sin \theta)$$

$$\frac{dr_2}{d\theta} = -2 \sin \frac{\pi}{3}$$

$$\frac{dr_2}{d\theta} = -\sqrt{3}$$

$$y'_1 = \frac{\sin \theta \frac{dr_2}{d\theta} + r_2 \cos \theta}{\cos \theta \frac{dr_2}{d\theta} - r_2 \sin \theta}$$

$$y'_1 = \frac{\left(\sin \frac{\pi}{3} \right) (-\sqrt{3}) + 3 \cos \frac{\pi}{3}}{\left(\cos \frac{\pi}{3} \right) (-\sqrt{3}) - 3 \sin \frac{\pi}{3}}$$

$$y'_1 = 0$$

$$\theta_1 = 0^\circ$$

You must get an equal value or r, since they intersect.

Therefore, the angle of intersection is:

$$\theta = |\theta_1 - \theta_2|$$

$$\theta = |30^\circ - 0^\circ|$$

$$\theta = 30^\circ$$

Alternative solution:

For the first curve:

$$\frac{d}{dx} (6 \cos X \cos X)_{X=\frac{\pi}{3}} = -5.196 \rightarrow \frac{dx}{d\theta}$$

$$\frac{d}{dx} (6 \cos X \sin X)_{X=\frac{\pi}{3}} = -3 \rightarrow \frac{dy}{d\theta}$$

Thus:

$$y'_1 = \frac{\left(\frac{dy}{d\theta} \right)}{\left(\frac{dx}{d\theta} \right)} = \frac{-3}{-5.196} = 0.577$$

$$\theta_1 = 30^\circ$$

For the second curve:

$$\frac{d}{dx} (2(1 + \cos \theta) \cos X)_{X=\frac{\pi}{3}} = -3.464 \rightarrow \frac{dx}{d\theta}$$

$$\frac{d}{dx} (2(1 + \cos \theta) \sin X)_{X=\frac{\pi}{3}} = 0 \rightarrow \frac{dy}{d\theta}$$

Thus:

$$y'_2 = \frac{\left(\frac{dy}{d\theta} \right)}{\left(\frac{dx}{d\theta} \right)} = \frac{0}{-3.464} = 0$$

$$y'_2 = 0^\circ$$

Therefore:

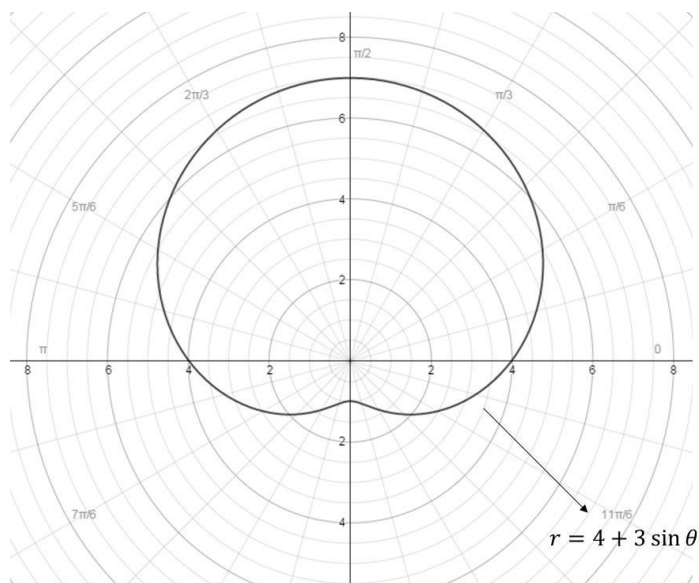
$$\theta = |\theta_1 - \theta_2|$$

$$\theta = |30^\circ - 0^\circ|$$

$$\theta = 30^\circ \text{ (ok!)}$$

50. Find the radius of curvature of the curve $r = 4 + 3\sin\theta$, at $\theta = 0$.

Solution:



From the formula:

$$\rho = \frac{(r^2 + r'^2)^{\frac{3}{2}}}{|r^2 + 2r'^2 - rr''|}$$

We have:

$$r = 4 + 3 \sin \theta$$

$$r = 4 + 3 \sin 0$$

$$r = 4; r^2 = 16$$

For the first derivative:

$$r' = 3 \cos \theta$$

$$r' = 3 \cos 0$$

$$r' = 3; r'^2 = 9$$

For the second derivative:

$$r'' = -3 \sin \theta$$

$$r'' = -3 \sin 0$$

$$r'' = 0$$

Therefore:

$$\rho = \frac{(16 + 9)^{\frac{3}{2}}}{|16 + 2(9) - 4(0)|}$$

$$\rho = \frac{125}{34}$$

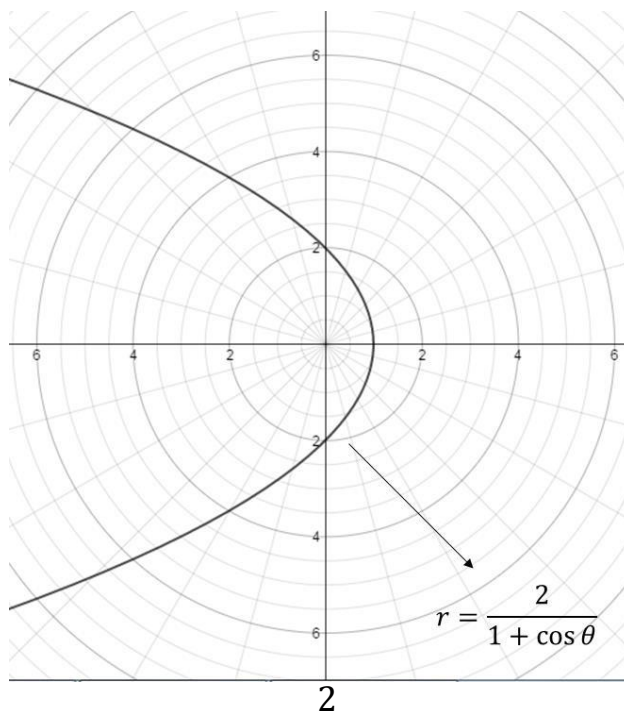
51. Find the radius of curvature of the curve $r = \frac{2}{1 + \cos \theta}$; $\theta = \frac{\pi}{2}$.

Solution:

From the formula:

$$\rho = \frac{(r^2 + r'^2)^{\frac{3}{2}}}{|r^2 + 2r'^2 - rr''|}$$

We have:



$$r = \frac{2}{1 + \cos \theta}$$

$$r = \frac{2}{1 + \cos \frac{\pi}{2}}$$

$$r = 2; r^2 = 4$$

For the first derivative:

$$r' = 2(-1)(1 + \cos \theta)^{-2}(-\sin \theta)$$

$$r' = \frac{2 \sin \theta}{(1 + \cos \theta)^2}$$

$$r' = \frac{2 \sin \frac{\pi}{2}}{(1 + \cos \frac{\pi}{2})^2}$$

$$r' = 2; r'^2 = 4$$

For the second derivative:

$$r'' = 2 \left[\frac{(1 + \cos \theta)^2 (\cos \theta) - \sin \theta (2)(1 + \cos \theta)(-\sin \theta)}{(1 + \cos \theta)^4} \right]$$

$$r'' = 2 \left[\frac{(1 + \cos \frac{\pi}{2})^2 (\cos \frac{\pi}{2}) + 2 \sin^2 \frac{\pi}{2} (1 + \cos \frac{\pi}{2})}{(1 + \cos \frac{\pi}{2})^4} \right]$$

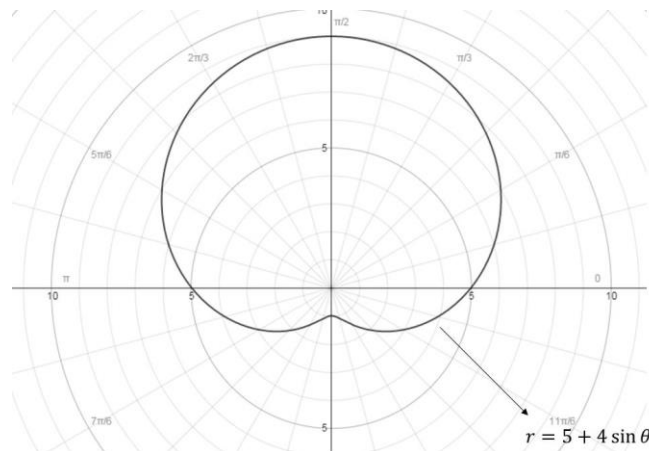
$$r'' = 4$$

Therefore:

$$\rho = \frac{(4 + 4)^{\frac{3}{2}}}{|4 + 2(4) - 2(4)|}$$

$$\rho = 4\sqrt{2}$$

Situation: For the curve $r = 5 + 4 \sin \theta$, $\frac{d\theta}{dt} = 2 \text{ rad/s}$, r is expressed in feet:



52. Find the radial component of the velocity at $\theta = \frac{\pi}{3}$.

Solution:

From the formula:

$$v_r = \frac{dr}{dt}$$

We have:

$$v_r = \frac{dr}{dt} = 4 \cos \theta \frac{d\theta}{dt}$$

$$v_r = 4 \cos \frac{\pi}{3} (2 \text{ rad/s})$$

$$v_r = \mathbf{4 \text{ ft/s}}$$

53. Find the transverse component of the velocity at $\theta = \frac{\pi}{3}$.

Solution:

From the formula:

$$v_\theta = r \frac{d\theta}{dt}$$

We have:

$$v_\theta = (5 + 4 \sin \theta) \left(\frac{2 \text{ rad}}{\text{s}} \right)$$

$$v_\theta = \left(5 + 4 \sin \frac{\pi}{3} \right) \left(2 \frac{\text{rad}}{\text{s}} \right)$$

$$v_\theta = \mathbf{10 + 4\sqrt{3} \approx 16.93 \text{ ft/s}}$$

54. Find the velocity of the particle when $\theta = \frac{\pi}{3}$.

Solution:

$$v^2 = v_r^2 + v_\theta^2$$

$$v^2 = 4^2 + 16.93^2$$

$$v = \mathbf{17.39 \text{ ft/s}}$$

55. Find the radial component of the acceleration at $\theta = \frac{\pi}{3}$.

Solution:

From the formula:

$$a_r = \frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2$$

We have:

$$\begin{aligned} \frac{dr}{dt} &= 4 \cos \theta \frac{d\theta}{dt} \\ \frac{d^2r}{dt^2} &= 4 \left[\cos \theta \frac{d^2\theta}{dt^2} + \frac{d\theta}{dt} \left(-\sin \theta \frac{d\theta}{dt} \right) \right] \\ \frac{d^2r}{dt^2} &= 4 \left[0 - \sin \frac{\pi}{3} (2)^2 \right] \\ \frac{d^2r}{dt^2} &= -8\sqrt{3} \end{aligned}$$

Therefore:

$$\begin{aligned} a_r &= -8\sqrt{3} - \left(5 + 4 \sin \frac{\pi}{3} \right) (2)^2 \\ a_r &= -20 - 16\sqrt{3} \approx -47.71 \text{ ft/s}^2 \end{aligned}$$

56. Find the transverse component of the acceleration at $\theta = \frac{\pi}{3}$.

Solution:

From the formula:

$$a_\theta = 2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2}$$

Therefore:

$$\begin{aligned} a_\theta &= 2(4)(2) + 0 \\ a_\theta &= 16 \text{ ft/s}^2 \end{aligned}$$

57. Find the acceleration of the particle when $\theta = \frac{\pi}{3}$.

Solution:

$$\begin{aligned} a^2 &= a_r^2 + a_\theta^2 \\ a^2 &= (-47.41)^2 + 16^2 \\ a &= 50.32 \text{ ft/s}^2 \end{aligned}$$

58. Find two positive numbers whose product is 64, and whose sum is a minimum.

Solution:

Let:

$$\begin{aligned} xy &= 64 \\ x &- \text{1st number} \\ y &- \text{2nd number} \end{aligned}$$

$$S = x + y$$

Solve y in terms of x:

$$y = \frac{64}{x}$$

Therefore:

$$S = x + \frac{64}{x}$$

$$\frac{dS}{dx} = 1 - \frac{64}{x^2}$$

Since the sum is a minimum, set the derivative equal to zero:

$$0 = 1 - \frac{64}{x^2}$$

$$x^2 - 64 = 0$$

$$x = 8$$

Solving for the value of y:

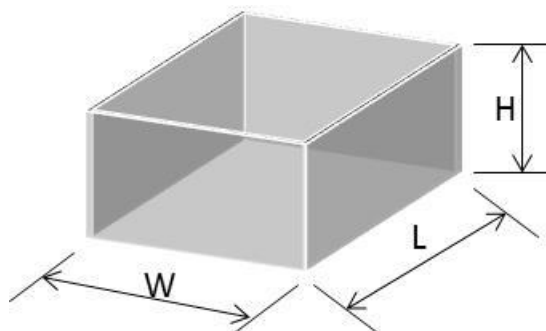
$$y = \frac{64}{8}$$

$$y = 8$$

59. A closed box, whose length is twice its width, is to have a surface of 192 sq. in. Find the dimensions of the box when the volume is maximum.

Solution:

Referring to the figure:



The surface area of a closed box is:

$$A = 2(LW + LH + WH)$$

Given that:

$$L = 2W$$

Substituting:

$$96 = (2W)W + (2W)H + WH$$

$$96 = 2W^2 + 2WH + WH$$

$$96 = 2W^2 + 3WH$$

Solving for the value of H:

$$3WH = 96 - 2W^2$$

$$H = \frac{32}{W} - \frac{2}{3}W$$

From the volume of rectangular prism:

$$V = LWH$$

$$V = (2W)WH$$

$$V = 2W^2 \left(\frac{32}{W} - \frac{2}{3}W \right)$$

$$V = 64W - \frac{4}{3}W^3$$

$$\frac{dV}{dW} = 64 - 4W^2$$

Setting the derivative equal to zero:

$$0 = 64 - 4W^2$$

$$W = 4 \text{ in}$$

Therefore:

$$L = 2(4)$$

$$L = 8 \text{ in}$$

$$H = \frac{32}{4} - \frac{2}{3}(4)$$

$$H = \frac{16}{3} \text{ in}$$

Therefore, the dimension is **4 in x 8 in x 5.33 in.**

60. Two vertices of a rectangle are on the positive x-axis. The other two vertices are on the lines $y = 4x$ and $y = -5x + 6$. What is the maximum possible area of the rectangle?

Solution:

Referring to the figure:

Let: $u =$ coordinate of the leftmost vertex, B:

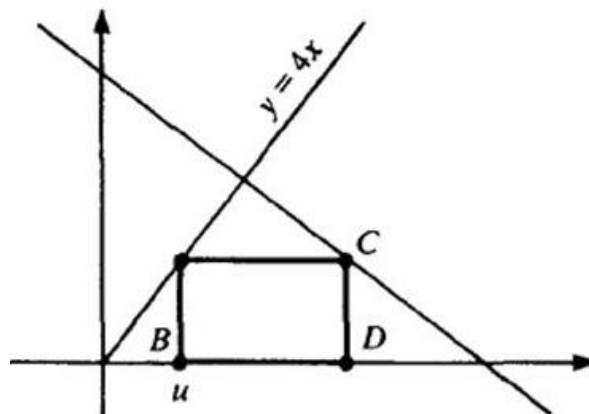
$$y = 4x$$

$$y = 4u$$

Substituting to the other equation:

$$4u = -5x + 6$$

$$x = \frac{6 - 4u}{5}$$



Thus, the base of the rectangle is:

$$b = \frac{6 - 4u}{5} - u = \frac{6 - 9u}{5}$$

Therefore:

$$A = bh$$

$$A = \left(\frac{6 - 9u}{5}\right)(4u)$$

$$A = \frac{4}{5}(6u - 9u^2)$$

$$\frac{dA}{du} = \frac{4}{5}(6 - 18u)$$

Setting the derivative equal to zero:

$$u = \frac{1}{3}$$

The area of the rectangle is:

$$A = \frac{4}{5} \left[6 \left(\frac{1}{3}\right) - 9 \left(\frac{1}{3}\right)^2 \right]$$

$$A = \frac{4}{5}$$

61. A triangular corner lot has perpendicular sides of lengths 120 feet and 160 feet. Find the dimensions of the largest rectangular building that can be constructed on a lot with sides parallel to the streets.

Solution:

Referring to the figure:

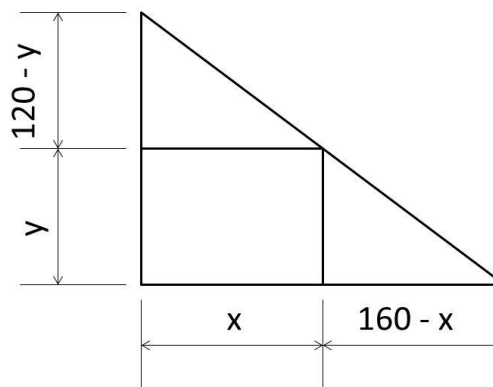
By similar triangles:

$$\frac{120}{160} = \frac{120 - y}{x}$$

$$\frac{3}{4} = \frac{120 - y}{x}$$

$$3x = 4(120 - y)$$

$$x = \frac{4}{3}(120 - y)$$



From the area of rectangle:

$$A = xy$$

$$A = \frac{4}{3}(120 - y)(y)$$

$$A = \frac{4}{3}(120y - y^2)$$

$$\frac{dA}{dy} = \frac{4}{3}(120 - 2y)$$

Setting the derivative equal to zero:

$$120 - 2y = 0$$

$$y = 60 \text{ ft}$$

$$\therefore x = 80 \text{ ft}$$

Therefore, the area is:

$$A = (60)(80)$$

$$A = 4800 \text{ ft}^2$$

62. Find the area of the largest rectangle with sides parallel to the coordinate axes which can be inscribed in the area bounded by the two parabolas $y = 26 - x^2$ and $y = x^2 + 2$.

Solution:

Referring to the figure:

$$A = 2x(y_2 - y_1)$$

$$A = 2x[(26 - x^2) - (x^2 + 2)]$$

$$A = 2x(24 - 2x^2)$$

$$A = 4x(12 - x^2)$$

$$A = 4(12x - x^3)$$

$$\frac{dA}{dx} = 4(12 - 3x^2)$$

Setting the derivative equal to zero:

$$3x^2 = 12$$

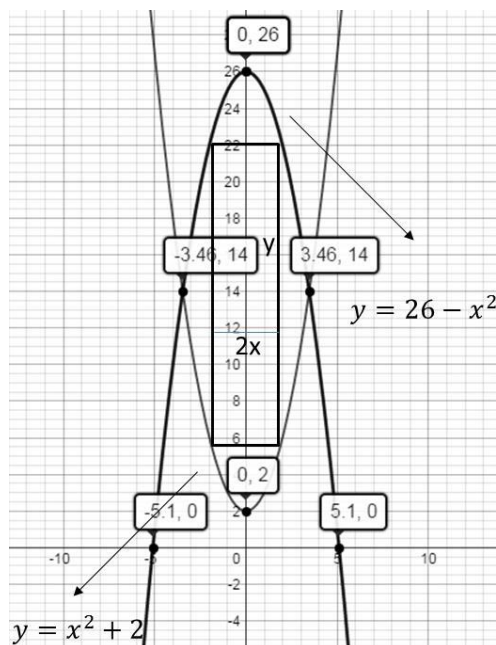
$$x^2 = 4$$

$$x = 2$$

Therefore, the area is:

$$A = 4[12(2) - 2^3]$$

$$A = 64 \text{ sq. units}$$



63. A concrete culvert has a cross-section consisting of a rectangle surmounted by a semicircle whose diameter is equal to the base of the rectangle. For a given perimeter, find the ratio of the height of the rectangle to its base if the area is a maximum.

Solution:

Referring to the figure:

$$P = 2h + b + \frac{\pi b}{2}$$

Solving for the value of h:

$$h = \frac{1}{2} \left[P - b \left(1 + \frac{\pi}{2} \right) \right]$$

For the area:

$$A = bh + \frac{\pi}{8} b^2$$

$$A = b \left\{ \frac{1}{2} \left[P - b \left(1 + \frac{\pi}{2} \right) \right] \right\} + \frac{\pi}{8} b^2$$

$$A = \frac{1}{2} \left[Pb - b^2 \left(1 + \frac{\pi}{2} \right) \right] + \frac{\pi}{8} b^2$$

$$\frac{dA}{db} = \frac{1}{2} \left[P - 2b \left(1 + \frac{\pi}{2} \right) \right] + \frac{\pi}{4} b$$

Setting the derivative equal to zero:

$$\frac{1}{2} \left[P - 2b \left(1 + \frac{\pi}{2} \right) \right] + \frac{\pi}{4} b = 0$$

$$\frac{1}{2} P - b \left(1 + \frac{\pi}{2} \right) + \frac{\pi}{4} b = 0$$

$$\frac{1}{2} P - \left(1 + \frac{\pi}{4} \right) b = 0$$

$$\left(1 + \frac{\pi}{4} \right) b = \frac{1}{2} P$$

$$b = \frac{P}{2 \left(1 + \frac{\pi}{4} \right)}$$

$$b = \frac{P}{2 + \frac{\pi}{2}} \cdot \frac{2}{2}$$

$$b = \frac{2P}{4 + \pi} \approx 0.28P$$

Therefore, the value of h is:

$$h = \frac{1}{2} \left[P - \left(\frac{2P}{4 + \pi} \right) \left(1 + \frac{\pi}{2} \right) \right]$$

$$h = \frac{1}{2} \left[P - \frac{2P}{4 + \pi} \left(\frac{2 + \pi}{2} \right) \right]$$

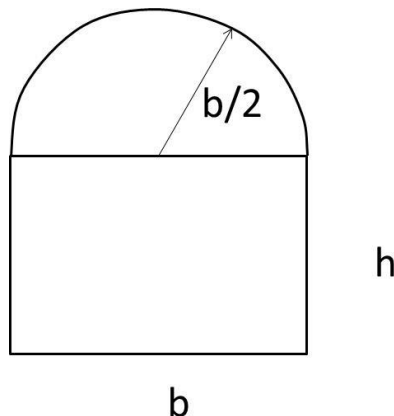
$$h = \frac{1}{2} \left[P - \frac{2 + \pi}{4 + \pi} P \right]$$

$$h = \frac{1}{2} P \left(\frac{\pi + 4}{\pi + 4} - \frac{2 + \pi}{4 + \pi} \right)$$

$$h = \frac{P}{\pi + 4} \approx 0.14P$$

The ratio of height to the base is:

$$\frac{h}{b} = \frac{\left(\frac{P}{\pi + 4} \right)}{\left(\frac{2P}{\pi + 4} \right)} = \frac{1}{2}$$



64. The area of a sector of a circle is 64 sq. cm. Find its radius if the perimeter is a maximum.

Solution:

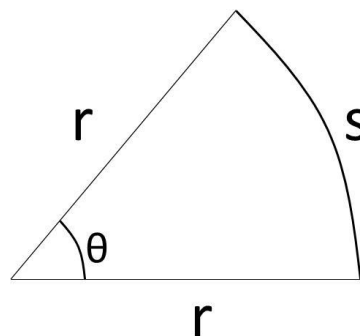
From the area of sector:

$$A = \frac{1}{2} r^2 \theta$$

$$64 = \frac{1}{2} r^2 \theta$$

$$128 = r^2 \theta$$

$$\theta = \frac{128}{r^2}$$



Referring to the figure:

$$P = 2r + r\theta$$

$$P = 2r + r \left(\frac{128}{r^2} \right)$$

$$P = 2r + \frac{128}{r}$$

$$\frac{dP}{dr} = 2 - \frac{128}{r^2}$$

Setting the derivative equal to zero:

$$0 = 2 - \frac{128}{r^2}$$

$$2r^2 = 128$$

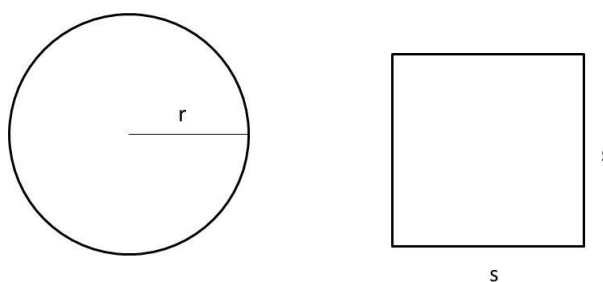
$$r^2 = 64$$

$$r = 8 \text{ cm}$$

65. A piece of wire of length $2\pi + 8$ cm is cut into two pieces. One piece is formed into a square and the other into a circle. Find the length of the circular wire so that the sum of the areas of the square and circle is a minimum.

Solution:

Referring to the figure:



$$P = 4s + 2\pi r$$

$$2\pi + 8 = 4s + 2\pi r$$

$$4s = 2\pi - 2\pi r + 8$$

$$4s = 2\pi(1 - r) + 8$$

$$s = \frac{\pi}{2}(1 - r) + 2$$

The sum of the areas of the two figures is:

$$A = s^2 + \pi r^2$$

Substituting the value of s :

$$A = \left[\frac{\pi}{2}(1 - r) + 2 \right]^2 + \pi r^2$$

$$\frac{dA}{dr} = 2 \left[\frac{\pi}{2}(1 - r) + 2 \right] \left(\frac{\pi}{2} \right) (-1) + 2\pi r$$

Setting the derivative equal to zero:

$$0 = -\pi \left[\frac{\pi}{2}(1-r) + 2 \right] + 2\pi r$$

$$2\pi r = \pi \left[\frac{\pi}{2}(1-r) + 2 \right]$$

$$2r = \frac{\pi}{2}(1-r) + 2$$

$$4r = \pi(1-r) + 4$$

$$4r = \pi - \pi r + 4$$

$$4r + \pi r = \pi + 4$$

$$r(\pi + 4) = \pi + 4$$

$$r = 1 \text{ cm}$$

Therefore, the other length is:

$$s = \frac{\pi}{2}(1-1) + 2$$

$$\mathbf{s = 2 \text{ cm}}$$

66. Find the shortest distance from the point (3, 0) to the parabola $y^2 = 4x$.

Solution:

By distance formula:

$$s^2 = (x-3)^2 + (y-0)^2$$

$$s^2 = (x-3)^2 + y^2$$

Substitute $y^2 = 4x$:

$$s^2 = (x-3)^2 + 4x$$

$$2s \frac{ds}{dx} = 2(x-3) + 4$$

Setting the derivative equal to zero:

$$0 = 2(x-3) + 4$$

$$2(x-3) = -4$$

$$x-3 = -2$$

$$x = 1$$

Substituting:

$$y^2 = 4(1)$$

$$y^2 = 4$$

$$y = 2$$

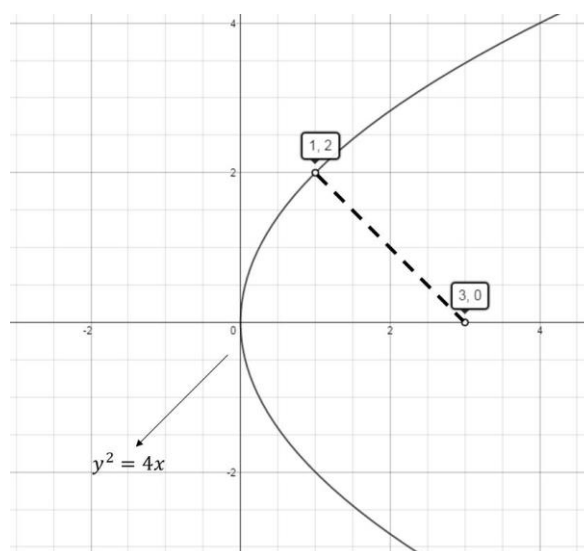
Therefore:

$$s^2 = (1-3)^2 + 2^2$$

$$s^2 = (-2)^2 + 2^2$$

$$s^2 = 8$$

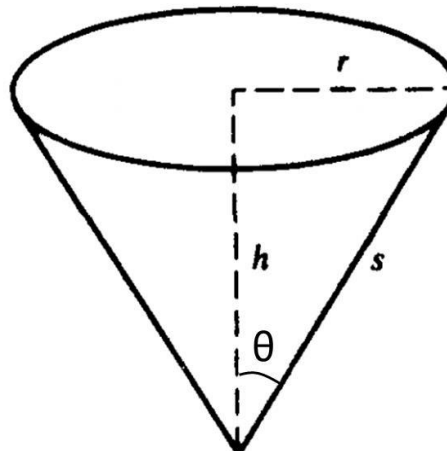
$$\mathbf{s = 2\sqrt{2}}$$



67. Find the volume of the largest right circular cone whose slant side is 10 cm. Solve using the functions of the angle θ .

Solution:

$$\begin{aligned}\sin \theta &= \frac{r}{10} \\ r &= 10 \sin \theta \\ \cos \theta &= \frac{h}{10} \\ h &= 10 \cos \theta\end{aligned}$$



Referring to the figure:

For the volume of the cone:

$$V = \frac{\pi}{3} r^2 h$$

Substituting the value of r and h in terms of θ :

$$\begin{aligned}V &= \frac{\pi}{3} (10 \sin \theta)^2 (10 \cos \theta) \\ V &= \frac{1000\pi}{3} \sin^2 \theta \cos \theta \\ \frac{dV}{d\theta} &= \frac{1000\pi}{3} [\sin^2 \theta (-\sin \theta) + \cos \theta (2 \sin \theta \cos \theta)]\end{aligned}$$

Setting the derivative to zero:

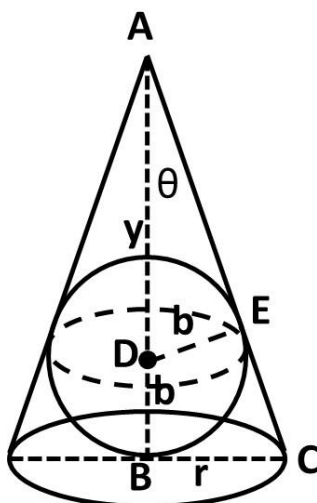
$$\begin{aligned}\sin^2 \theta &= 2 \cos^2 \theta \\ \frac{\sin^2 \theta}{\cos^2 \theta} &= 2 \\ \tan^2 \theta &= 2 \\ \tan \theta &= \sqrt{2} \\ \theta &= \tan^{-1} \sqrt{2} \\ \theta &= 54.74^\circ\end{aligned}$$

Therefore, the volume is:

$$\begin{aligned}V &= \frac{\pi}{3} (10 \sin 54.74^\circ)^2 (10 \cos 54.74^\circ) \\ V &= 403.07 \text{ cm}^3\end{aligned}$$

68. Find the volume of the right circular cone of greatest volume which can be circumscribed about a sphere of radius 10 cm. Solve using the function of the angle θ .

Solution:



Referring to the figure: ($b = 10$)

$$\sin \theta = \frac{b}{y}$$

$$\sin \theta = \frac{10}{y}$$

$$y = 10 \csc \theta$$

$$\tan \theta = \frac{r}{y + b}$$

$$\tan \theta = \frac{r}{10 \csc \theta + 10}$$

$$r = 10 \tan \theta (1 + \csc \theta)$$

$$r = 10(\tan \theta + \sec \theta)$$

From the formula of volume of a cone:

$$V = \frac{\pi}{3} r^2 h$$

$$V = \frac{\pi}{3} [10(\tan \theta + \sec \theta)]^2 (10)(1 + \csc \theta)$$

$$V = \frac{1000\pi}{3} (\tan \theta + \sec \theta)^2 (1 + \csc \theta)$$

$$\frac{dV}{d\theta} = \frac{1000\pi}{3} [(\tan \theta + \sec \theta)^2 (-\csc \theta \cot \theta)$$

$$+ (1 + \csc \theta)(2)(\sec \theta + \tan \theta)(\sec \theta \tan \theta + \sec^2 \theta)]$$

Setting the derivative equal to zero and simplifying:

$$\csc \theta \cot \theta (\tan \theta + \sec \theta)^2 = 2 \sec \theta (1 + \csc \theta)(\sec \theta + \tan \theta)^2$$

$$\csc \theta \cot \theta = 2 \sec \theta (1 + \csc \theta)$$

$$\frac{1}{\sin \theta} \cot \theta = \frac{2}{\cos \theta} \left(1 + \frac{1}{\sin \theta}\right)$$

$$\frac{1}{\sin \theta} \cot \theta = \frac{2}{\cos \theta} \left(\frac{\sin \theta + 1}{\sin \theta}\right)$$

$$\cot \theta = \frac{2}{\cos \theta} (1 + \sin \theta)$$

$$\frac{\cos \theta}{\sin \theta} = \frac{2}{\cos \theta} (1 + \sin \theta)$$

$$\cos^2 \theta = 2 \sin \theta (1 + \sin \theta)$$

$$\cos^2 \theta = 2 \sin \theta + 2 \sin^2 \theta$$

$$1 - \sin^2 \theta = 2 \sin \theta + 2 \sin^2 \theta$$

$$3 \sin^2 \theta + 2 \sin \theta - 1 = 0$$

$$(3 \sin \theta - 1)(\sin \theta + 1) = 0$$

$$\sin \theta = \frac{1}{3}; \sin \theta = -1$$

Adopt $\sin \theta = \frac{1}{3}$.

$$\therefore \csc \theta = 3; \tan \theta = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}; \sec \theta = \frac{3\sqrt{2}}{4}$$

Therefore:

$$V = \frac{1000\pi}{3} (\tan \theta + \sec \theta)^2 (1 + \csc \theta)$$

$$V = \frac{1000\pi}{3} \left(\frac{\sqrt{2}}{4} + \frac{3\sqrt{2}}{4} \right)^2 (1 + 3)$$

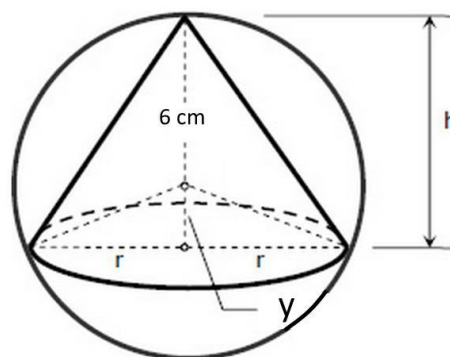
$$V = \frac{1000\pi}{3} (\sqrt{2})^2 (4)$$

$$V = \frac{8000\pi}{3} \text{ cm}^3$$

69. Find the volume of the right circular cone of greatest volume which can be inscribed in a sphere of radius 6 cm.

Solution:

Referring to the figure:



By Pythagorean Theorem:

$$r^2 + y^2 = 6^2$$

$$r^2 = 6^2 - y^2$$

From the volume of a cone:

$$V = \frac{\pi}{3} r^2 h$$

$$V = \frac{\pi}{3} (6^2 - y^2)(6 + y)$$

$$\frac{dV}{dy} = \frac{\pi}{3} [(6^2 - y^2) + (6 + y)(-2y)]$$

Setting the derivative equal to zero:

$$6^2 - y^2 = 12y + 2y^2$$

$$3y^2 + 12y - 36 = 0$$

$$y^2 + 4y - 12 = 0$$

$$y = 2; y = -6$$

Adopt $y = 2$ cm.

Therefore:

$$V = \frac{\pi}{3} (6^2 - 2^2)(6 + 2)$$

$$V = \frac{256\pi}{3} \text{ cm}^3$$

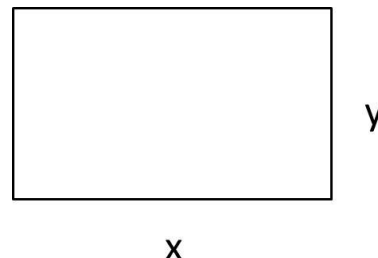
70. A lot 625 sq. m. in area is to be fenced on all four sides. Its dimensions require the least amount of fencing. If the fence costs P 50.00 per linear meter, what is the total cost?

Solution:

Referring to the figure:

$$A = xy = 625$$

$$y = \frac{625}{x}$$



Therefore, the perimeter is:

$$P = 2(x + y)$$

$$P = 2\left(x + \frac{625}{x}\right)$$

$$\frac{dP}{dx} = 2\left(1 - \frac{625}{x^2}\right)$$

Setting the derivative equal to zero:

$$1 - \frac{625}{x^2} = 0$$

$$x^2 = 625$$

$$x = 25 \text{ m}$$

$$\therefore y = 25 \text{ m}$$

Thus:

$$P = 2(25 + 25)$$

$$P = 100 \text{ m}$$

Therefore:

$$\text{Cost} = (100 \text{ m})\left(\frac{\text{P}50}{\text{m}}\right)$$

$$\text{Cost} = \text{P}5000$$

71. A right circular cylinder having a volume of 1200 cu. cm is to be lined with gold foil costing P 1.00 per sq. cm. on its curved surface and with silver foil costing P 0.60 per sq. cm. at the top and bottom. Find its height for minimum cost.

Solution:

Referring to the figure:

$$V = \pi r^2 h = 1200 \text{ cm}^3$$

$$h = \frac{1200}{\pi r^2}$$

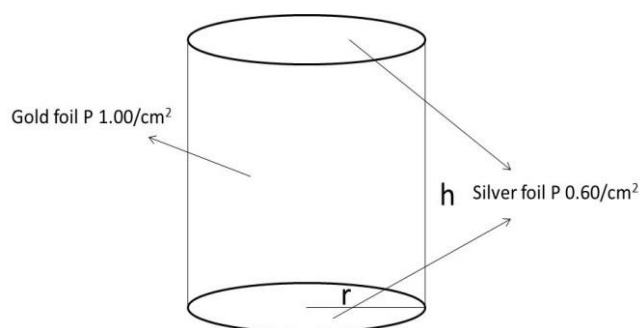
$$C = 2\pi r h(1) + 2\pi r^2(0.6)$$

$$C = 2\pi r h + 1.2\pi r^2$$

$$C = 2\pi r \left(\frac{1200}{\pi r^2}\right) + 1.2\pi r^2$$

$$C = \frac{2400}{r} + 1.2\pi r^2$$

$$\frac{dC}{dr} = -\frac{2400}{r^2} + 2.4\pi r$$



Setting the derivative equal to zero:

$$2.4\pi r = \frac{2400}{r^2}$$

$$2.4\pi r^3 = 2400$$

$$r^3 = \frac{1000}{\pi}$$

$$r = 6.827 \text{ cm}$$

$$\therefore h = 8.193 \text{ cm}$$

72. A gasoline station selling x gallons of fuel per month has fixed cost of P 2500 and variable costs of $0.90x$. The demand function is $1.50 - 0.00002x$ and the station's capacity allows no more than 20,000 gallons to be sold per month. Find the maximum profit.

Solution:

$$\text{total income} = x(1.50 - 0.00002x)$$

$$\text{profit, } G = x(1.50 - 0.00002x) - (2500 + 0.90x)$$

$$G = 0.60x - 0.00002x^2 - 2500$$

Differentiating:

$$\frac{dG}{dx} = 0.60 - 0.00004x$$

Setting the derivative equal to zero:

$$0.00004x = 0.60$$

$$x = 15000$$

Therefore, the profit is:

$$G = 0.60(15000) - 0.00002(15000)^2 - 2500$$

$$\mathbf{G = P 2000}$$

INTEGRAL CALCULUS

- Find the area in the first quadrant bounded by the coordinate axes and the parabolic arc $\sqrt{x} + \sqrt{y} = 1$.

A. $\frac{1}{3}$	C. $\frac{1}{4}$
B. $\frac{1}{6}$	D. $\frac{1}{12}$
- Find the total area between the cubic $y = 2x^3 - 3x^2 - 12x$, the x-axis, and its maximum and minimum ordinates.

A. 30.622	C. 8.360
B. 41.983	D. 50.344
- Find the area between the curves $x^2 = 2ay$ and $x^2 = 4ay - a^2$.

A. $\frac{a^2}{3}$	C. $\frac{a^2}{4}$
B. $\frac{a^2}{6}$	D. $\frac{a^2}{12}$
- Find the area of the ellipse given the equation $x = a \cos \phi$ and $y = b \sin \phi$.

A. $\pi a^2 b$	C. πab^2
B. πab	D. $2\pi a^2 b^2$
- Find the area enclosed by the curve $r = a \cos 3\theta$.

A. $\pi a^2/3$	C. $\pi a^2/4$
B. $\pi a^2/6$	D. $\pi a^2/12$
- The area under one arch of the sine curve revolves about the x-axis. Find the volume generated.

A. $\pi^2/2$	C. $\pi/2$
B. $\pi^2/6$	D. $2\pi^2/3$
- The area bounded by the parabola $ay = x^2$, the x-axis, and the line $x = b$ is revolved about the line $x = b$. Find the volume generated.

A. $\pi b^4/6a$	C. $\pi a^4/6b$
B. $\pi ab^4/12$	D. $\pi^2 b^3/6a$
- Find the volume generated by revolving about the x-axis the area in the second quadrant under the curve $y = e^x$.

A. $\pi^2/2$	C. $\pi/2$
B. $\pi^2/6$	D. $2\pi^2/3$
- Find the volume generated by revolving the area bounded the curve $x^3 y = 1$, $y = 0$, $x = 1$, $x = 2$; about $x = 1$.

A. $\pi^2/2$	C. $\pi/2$
B. $\pi/4$	D. $2\pi^2/3$
- Find the volume generated by revolving the area bounded by the curve outside $y = x^2$, and between $y = 2x - 1$ and $y = x + 2$; about y-axis.

A. $256\pi/15$	C. $31\pi/30$
B. $7\pi/2$	D. $227\pi/60$

INTEGRAL CALCULUS SOLUTIONS

Area of Plane Figures

1. Find the area in the first quadrant bounded by the coordinate axes and the parabolic arc $\sqrt{x} + \sqrt{y} = 1$.

Solution:

Referring to the figure:

We have:

$$dA = y \, dx$$

$$A = \int y \, dx$$

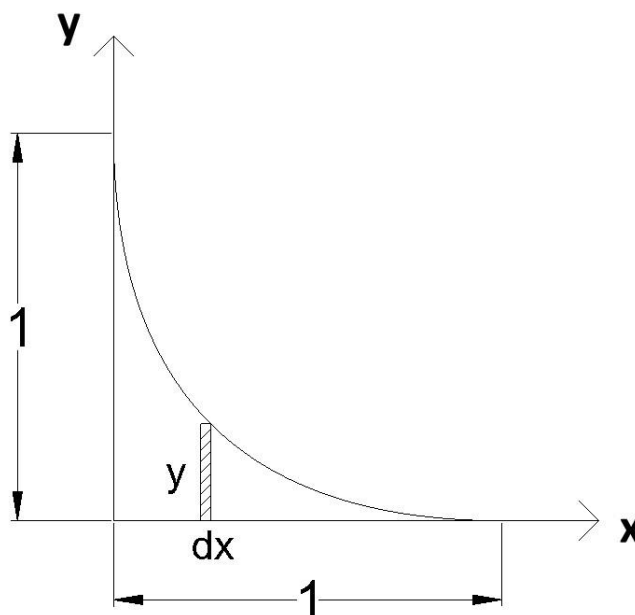
$$\sqrt{y} = 1 - \sqrt{x}$$

$$y = (1 - \sqrt{x})^2$$

Therefore:

$$A = \int_0^1 (1 - \sqrt{x})^2 \, dx$$

$$A = \frac{1}{6} \text{ square unit}$$



2. Find the total area between the cubic $y = 2x^3 - 3x^2 - 12x$, the x-axis, and its maximum and minimum ordinates.

Solution:

Solve for the roots:

$$y = 2x^3 - 3x^2 - 12x$$

Using MODE 5-4:

$$x_1 = \frac{3 + \sqrt{105}}{4} = 3.31$$

$$x_2 = \frac{3 - \sqrt{105}}{4} = -1.81$$

$$x_3 = 0$$

Then, solve for y' and set $y' = 0$:

$$y' = 6x^2 - 6x - 12$$

$$0 = 6x^2 - 6x - 12$$

Using MODE 5-3:

$$x_1 = 2; y = -20$$

$$x_2 = -1; y = 7$$

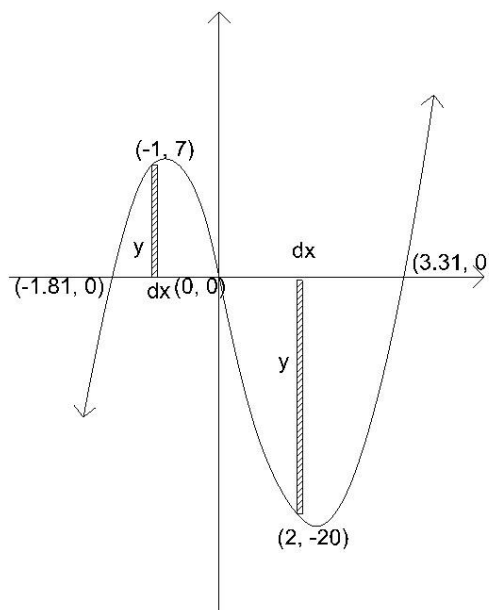
Then plot the curve:

Referring to the figure:

$$dA = y \, dx$$

$$A_1 = \int_{-1.81}^0 (2x^3 - 3x^2 - 12x) \, dx$$

$$A_1 = 8.360$$



$$A_2 = \int_0^{3.31} [0 - (2x^3 - 3x^2 - 12x)] dx$$

$$A_2 = 41.983$$

Therefore, we have:

$$A_T = A_1 + A_2$$

$$A_T = 8.360 + 41.983$$

$$A_T = \frac{1611}{32} = \mathbf{50.344}$$

3. Find the area between the curves $x^2 = 2ay$ and $x^2 = 4ay - a^2$.

Solution:

Solving for the point(s) of intersection:

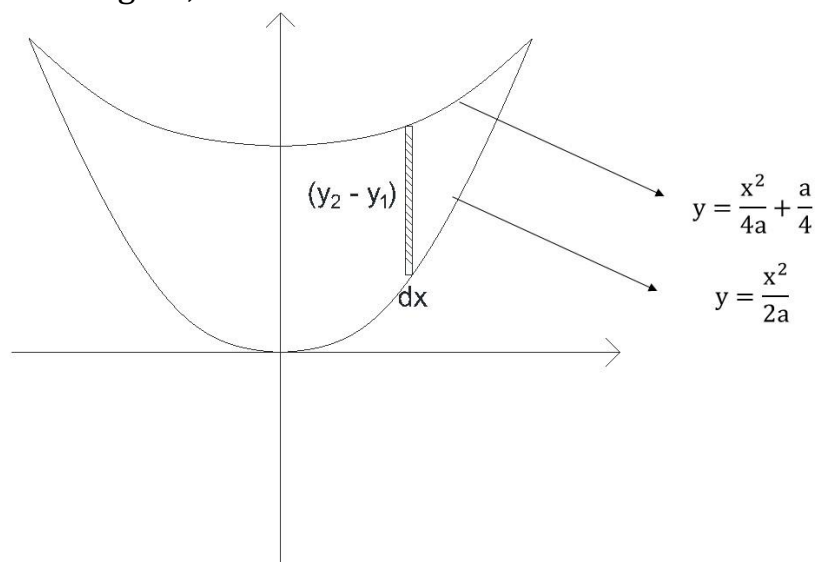
$$2ay = 4ay - a^2$$

$$-2ay = -a^2$$

$$y = \frac{a}{2}$$

$$\therefore x = \pm a$$

Referring to the figure, we have:



$$dA = (y_2 - y_1)dx$$

$$A = \int (y_2 - y_1)dx$$

By symmetry:

$$A = 2 \int_0^a \left[\left(\frac{x^2}{4a} + \frac{a}{4} \right) - \frac{x^2}{2a} \right] dx$$

$$A = 2 \int_0^a \left(\frac{a}{4} - \frac{x^2}{4a} \right) dx$$

$$A = 2 \left[\frac{a}{4}x - \frac{x^3}{12a} \right]_0^a$$

$$A = 2 \left[\frac{a^2}{4} - \frac{a^3}{12a} \right]$$

$$A = \frac{a^2}{3}$$

4. Find the area of the ellipse given the equation $x = a \cos \varphi$ and $y = b \sin \varphi$.

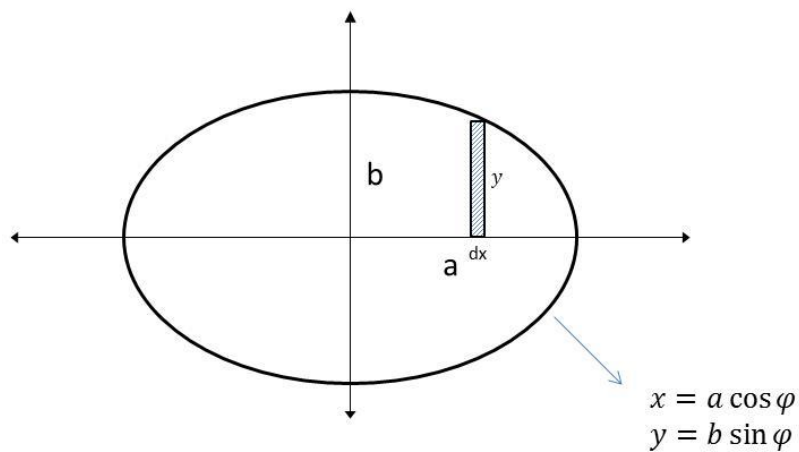
Solution:

When $x = 0$:

$$\begin{aligned} x &= a \cos \varphi \\ 0 &= a \cos \varphi \\ \cos \varphi &= 0 \\ \varphi &= \frac{\pi}{2} \end{aligned}$$

When $x = a$:

$$\begin{aligned} x &= a \cos \varphi \\ a &= a \cos \varphi \\ \cos \varphi &= 1 \\ \varphi &= 0 \end{aligned}$$



We have:

$$\begin{aligned} dA &= 4y \, dx \\ y &= b \sin \varphi \\ x &= a \cos \varphi \\ dx &= -a \sin \varphi \, d\varphi \end{aligned}$$

Substituting:

$$\begin{aligned} A &= 4 \int_{\frac{\pi}{2}}^0 (b \sin \varphi)(-a \sin \varphi \, d\varphi) \\ A &= 4ab \int_0^{\frac{\pi}{2}} \sin^2 \varphi \, d\varphi \\ \mathbf{A} &= \mathbf{\pi ab} \end{aligned}$$

5. Find the area enclosed by the curve $r = a \cos 3\theta$.

Solution:

r	a	0	$-\frac{\sqrt{2}}{2}a$	-a	0	a	$\frac{\sqrt{2}}{2}a$	0	-a
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π

Plotting the points:

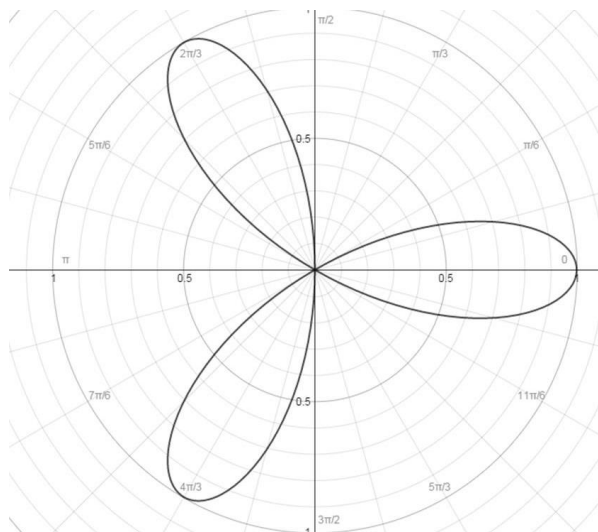
$$dA = \frac{1}{2} r^2 d\theta$$

$$A = \frac{1}{2} \int r^2 d\theta$$

$$A = 6 \left[\frac{1}{2} \int_0^{\frac{\pi}{6}} (a \cos 3\theta)^2 d\theta \right]$$

$$A = 6 \left[\frac{1}{2} a^2 \int_0^{\frac{\pi}{6}} \cos^2 3\theta d\theta \right]$$

$$A = \frac{\pi}{4} a^2$$



Solids of Revolution

6. The area under one arch of the sine curve revolves about the x-axis. Find the volume generated.

Solution:

Referring to the figure:

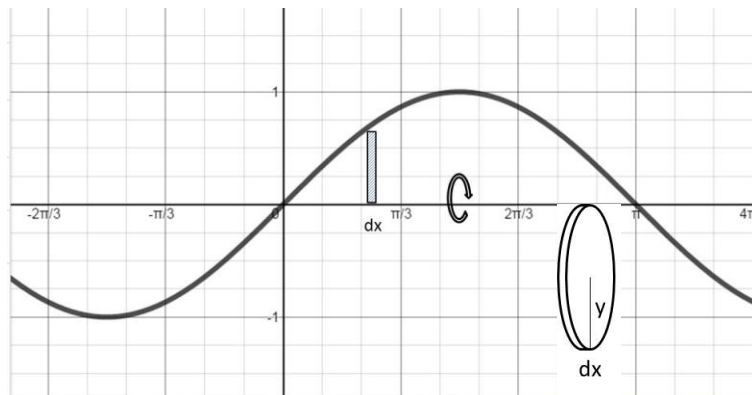
We have:

$$dV = \pi y^2 dx$$

$$V = \pi \int y^2 dx$$

$$V = \pi \int_0^{\pi} (\sin x)^2 dx$$

$$V = \frac{\pi^2}{2} \text{ cubic units}$$



7. The area bounded by the parabola $ay = x^2$, the x-axis, and the line $x = b$ is revolved about the line $x = b$. Find the volume generated.

Solution:

Referring to the figure:

We have:

$$dV = 2\pi(b - x)y dx$$

$$V = 2\pi \int (b - x)y dx$$

$$V = 2\pi \int_0^b (b - x) \left(\frac{x^2}{a} \right) dx$$

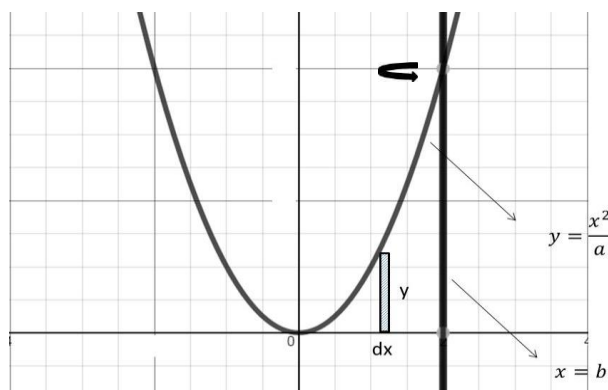
$$V = 2\pi \int_0^b \left(\frac{b}{a} x^2 - \frac{x^3}{a} \right) dx$$

$$V = 2\pi \left[\frac{b}{a} \cdot \frac{x^3}{3} - \frac{x^4}{4a} \right]_0^b$$

$$V = 2\pi \left[\frac{b}{a} \cdot \frac{b^3}{3} - \frac{b^4}{4a} \right]$$

$$V = 2\pi \left[\frac{b^4}{3a} - \frac{b^4}{4a} \right]$$

$$V = \frac{\pi b^4}{6a} \text{ cubic units}$$



8. Find the volume generated by revolving about the x-axis the area in the second quadrant under the curve $y = e^x$.

Solution:

Referring to the figure:

We have:

$$dV = \pi y^2 dx$$

$$V = \pi \int y^2 dx$$

$$V = \pi \int_{-\infty}^0 (e^x)^2 dx$$

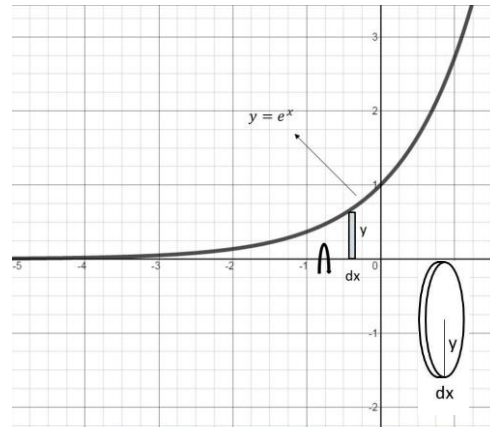
$$V = \pi \cdot \lim_{b \rightarrow -\infty} \int_b^0 e^{2x} dx$$

$$V = \pi \cdot \lim_{b \rightarrow -\infty} \left[\frac{1}{2} e^{2x} \right]_b^0$$

$$V = \pi \cdot \lim_{b \rightarrow -\infty} \left(\frac{1}{2} e^0 - \frac{1}{2} e^{2b} \right)$$

$$V = \pi \left(\frac{1}{2} \right)$$

$$\mathbf{V = \frac{\pi}{2} \text{ cubic units}}$$



9. Find the volume generated by revolving the area bounded the curve $x^3y = 1$, $y = 0$, $x = 1$, $x = 2$; about $x = 1$.

Solution:

Referring to the figure:

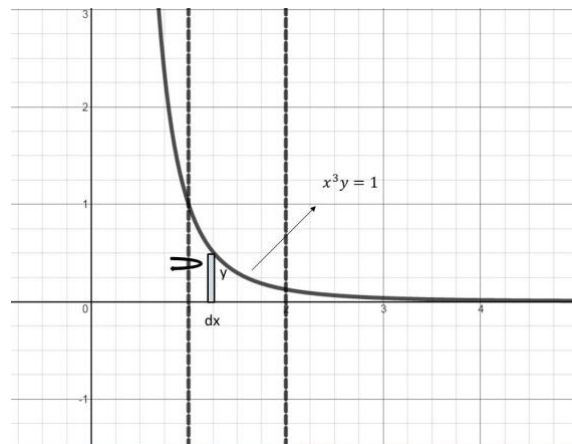
We have:

$$dV = 2\pi(x - 1)y dx$$

$$V = 2\pi \int (x - 1)y dx$$

$$V = 2\pi \int_1^2 (x - 1)(x^{-3}) dx$$

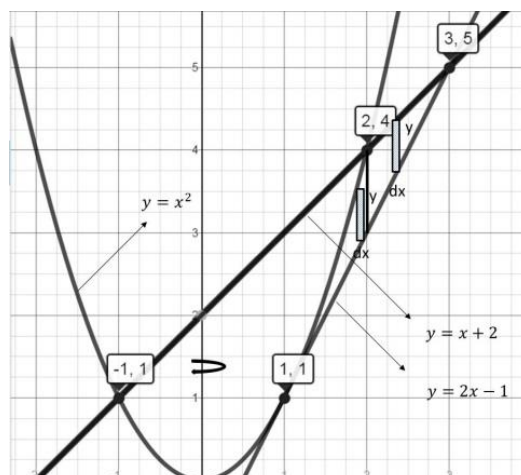
$$\mathbf{V = \frac{\pi}{4} \text{ cubic units}}$$



10. Find the volume generated by revolving the area bounded by the curve outside $y = x^2$, and between $y = 2x - 1$ and $y = x + 2$; about y-axis.

Solution:

Solve for the point of intersection:



$$x^2 = 2x - 1$$

$$x^2 - 2x + 1 = 0$$

$$x = 1; y = 1$$

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$x = 2; y = 4$$

$$x = -1; y = 1$$

$$2x - 1 = x + 2$$

$$x = 3$$

Then plot the graph:

Referring to the figure, we have:

$$dV = \pi x^2 y \, dx$$

$$V = 2\pi \int xy \, dx$$

$$V_1 = 2\pi \int x(y_2 - y_1) \, dx$$

$$V_1 = 2\pi \int_1^2 x[x^2 - (2x - 1)] \, dx$$

$$V_1 = \frac{7}{6}\pi$$

$$V_2 = \pi \int x(y_2 - y_3) \, dx$$

$$V_2 = \pi \int_2^3 x[(x + 2) - (2x - 1)] \, dx$$

$$V_2 = \frac{7}{3}\pi$$

Therefore,

$$V_T = V_1 + V_2$$

$$V_T = \frac{7}{6}\pi + \frac{7}{3}\pi$$

$$V_T = \frac{7}{2}\pi \text{ cubic units}$$

Arc Length

11. Find the length of the curve $y = \ln \cos x$ from $x = 0$ to $x = \frac{\pi}{4}$.

Solution:

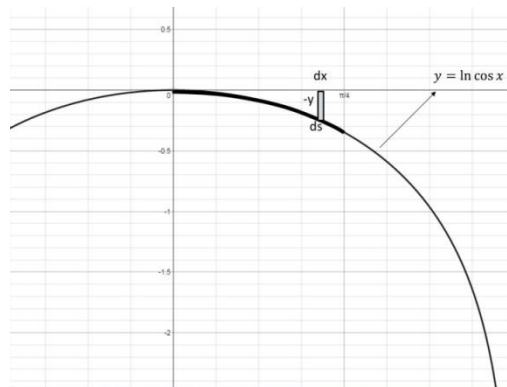
We have:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$y = \ln \cos x$$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$$



Therefore:

$$s = \int_0^{\pi/4} \sqrt{1 + (-\tan x)^2} dx$$

$$s = \int_0^{\pi/4} \sqrt{1 + \tan^2 x} dx$$

$$s = \int_0^{\pi/4} \sec x dx$$

$$s = \ln(1 + \sqrt{2})$$

$$s = \sinh^{-1} 1$$

$$s = \mathbf{0.8814}$$

12. Find the length of the curve of one branch of the curve $9y^2 = 4x^3$ from $x = 0$ to $x = 3$.

Solution:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

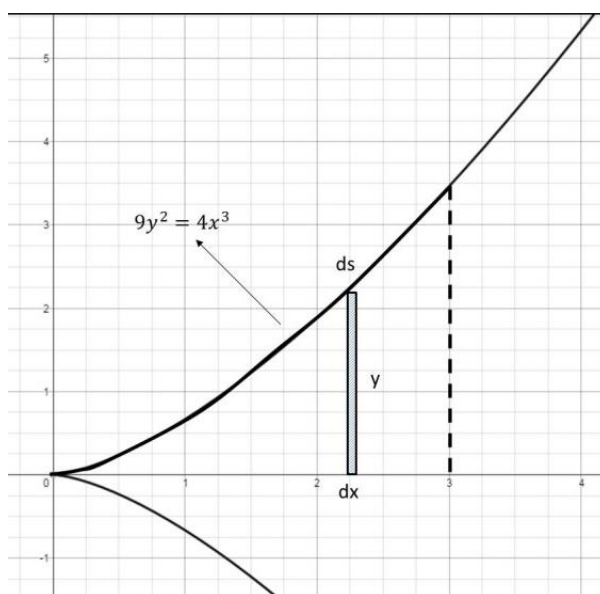
$$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$y^2 = \frac{4}{9}x^3$$

$$y = \frac{2}{3}x^{3/2}$$

$$y' = \frac{2}{3} \cdot \frac{3}{2} x^{1/2}$$

$$y' = x^{1/2}$$



Therefore,

$$s = \int_0^3 \sqrt{1 + x} dx$$

$$s = \mathbf{\frac{14}{3}}$$

13. Find the length of the curve whose parametric equations are $x = at^2$ and $y = at^3$, from $t = 0$ and $t = \sqrt{5}$.

Solution:

We have:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\frac{dx}{dt} = 2at; \frac{dy}{dt} = 3at^2$$

$$\left(\frac{dx}{dt}\right)^2 = 4a^2t^2; \left(\frac{dy}{dt}\right)^2 = 9a^2t^4$$

When $t = 0$:

$$x = a(0)^2 \rightarrow x = 0; y = a(0)^3 \rightarrow y = 0$$

When $t = \sqrt{5}$:

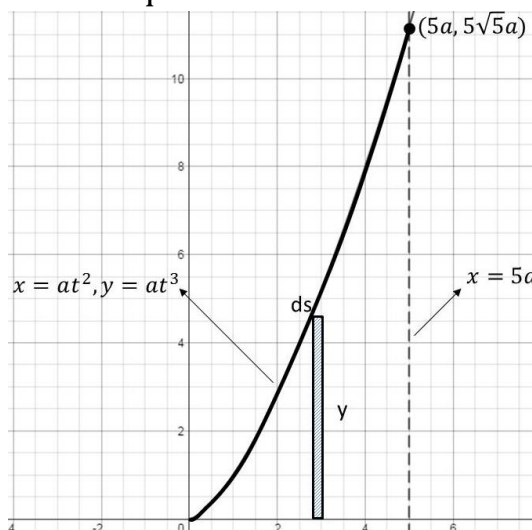
$$x = a(\sqrt{5})^2 \rightarrow x = 5a; y = a(\sqrt{5})^3 \rightarrow y = 5\sqrt{5}a$$

Therefore:

$$s = \int_0^{\sqrt{5}} \sqrt{4a^2t^2 + 9a^2t^4} dt$$

$$s = a \int_0^{\sqrt{5}} \sqrt{4t^2 + 9t^4} dt$$

$$s = \frac{335}{27} a$$



14. Find the length of the curve $y = \arcsin(e^{-x})$, from $x = 0$ to $x = \ln\left(\frac{5}{4}\right)$.

Solution:

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$s = \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$y = \arcsin(e^{-x})$$

When $x = 0$:

$$y = \arcsin(e^0) = \frac{\pi}{2}$$

When $x = \ln 1.25$:

$$y = \arcsin[e^{(-\ln 1.25)}] = 0.9273$$

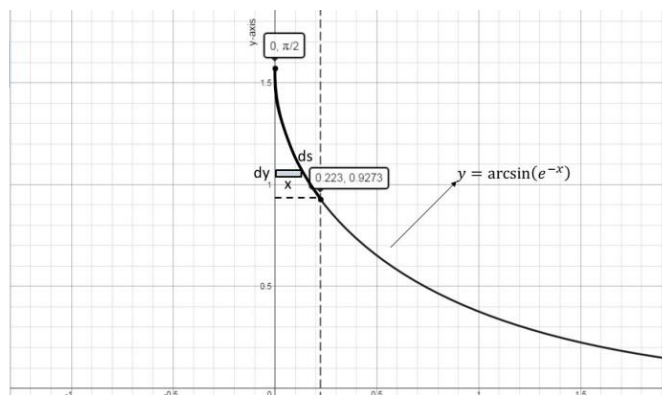
Since the function is decreasing, we must interchange the position of the limits for integration to avoid negative result.

$$\sin y = e^{-x}$$

$$\ln(\sin y) = -x$$

$$x = -\ln(\sin y)$$

$$\frac{dx}{dy} = -\frac{\cos y}{\sin y} = -\cot y$$



Therefore:

$$s = \int_{0.9273}^{\frac{\pi}{2}} \sqrt{1 + (-\cot y)^2} dy$$

$$s = \int_{0.9273}^{\frac{\pi}{2}} \sqrt{1 + \cot^2 y} dy$$

$$s = \int_{0.9273}^{\frac{\pi}{2}} \csc y dy$$

$$s = \mathbf{0.6931}$$

15. Find the total length of the cardioid $r = a(1 + \cos \theta)$.

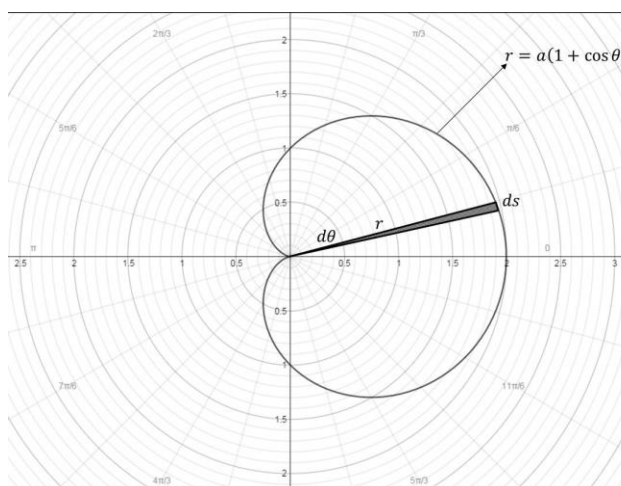
Solution:

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$s = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$r = a(1 + \cos \theta)$$

$$\frac{dr}{d\theta} = -a \sin \theta$$



Therefore:

By symmetry:

$$s = 2 \int_0^{\pi} \sqrt{[a(1 + \cos \theta)]^2 + (-a \sin \theta)^2} d\theta$$

$$s = 2 \int_0^{\pi} \sqrt{a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta} d\theta$$

$$s = 2a \int_0^{\pi} \sqrt{(1 + \cos \theta)^2 + \sin^2 \theta} d\theta$$

$$s = \mathbf{8a}$$

Surface of Revolution

16. Given that $x = a \cos \theta$ and $y = a \sin \theta$, find the surface area when revolved about the y-axis.

Solution:

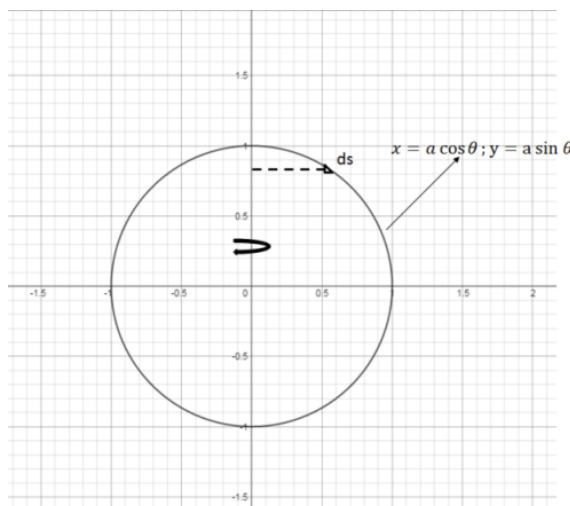
$$dA = 2\pi x ds$$

$$A = 2\pi \int x ds$$

$$A = 2\pi \int x \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

We have:

$$\frac{dx}{d\theta} = -a \sin \theta; \frac{dy}{d\theta} = a \cos \theta$$



Thus,

$$A = 2 \left[2\pi \int_0^{\frac{\pi}{2}} (a \cos \theta) \sqrt{(-a \sin \theta)^2 + (a \cos \theta)^2} d\theta \right]$$

$$A = 2 \left[2\pi \int_0^{\frac{\pi}{2}} a^2 \cos \theta d\theta \right]$$

$$A = 4\pi a^2$$

17. Find the surface area generated by revolving one arch of the cosine curve about the x-axis.

Solution:

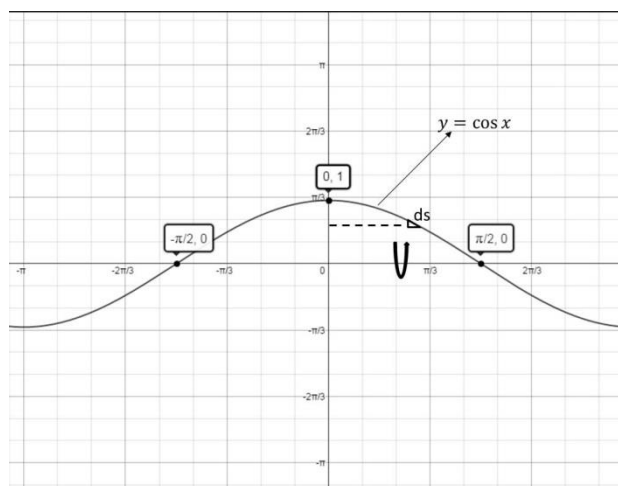
$$dA = 2\pi y ds$$

$$A = 2\pi \int y ds$$

$$A = 2\pi \int y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$y = \cos x$$

$$\frac{dy}{dx} = -\sin x$$



Therefore:

$$A = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \sqrt{1 + (-\sin x)^2} dx$$

$$A = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \sqrt{1 + \sin^2 x} dx$$

$$A = 14.42$$

18. Find the surface area generated by revolving $y = x^2$ from $x = 0$ to $x = 2$, about the x-axis.

Solution:

$$dA = 2\pi y ds$$

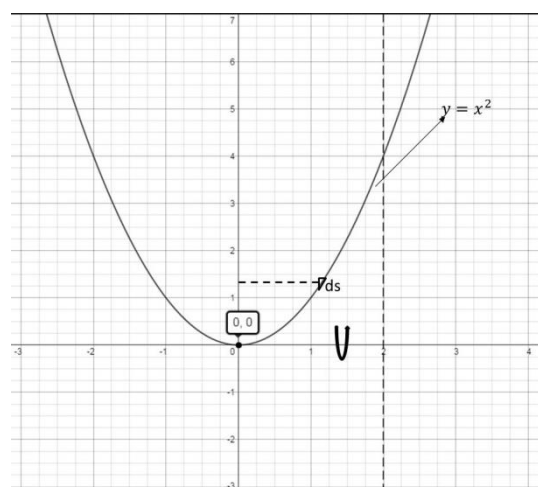
$$A = 2\pi \int y ds$$

$$A = 2\pi \int y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

$$\left(\frac{dy}{dx}\right)^2 = 4x^2$$



Therefore:

$$A = 2\pi \int_0^2 x^2 \sqrt{1 + 4x^2} dx$$

$$A = 53.226$$

19. Find the surface area generated by revolving an arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ about its base.

Solution:

Referring to the figure:

$$dA = 2\pi y ds$$

$$A = 2\pi \int y ds$$

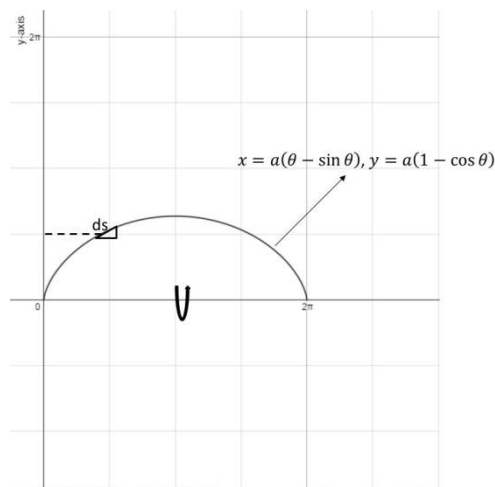
$$A = 2\pi \int y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

$$x = a(\theta - \sin \theta)$$

$$\frac{dx}{d\theta} = a(1 - \cos \theta)$$

$$y = a(1 - \cos \theta)$$

$$\frac{dy}{d\theta} = a(\sin \theta)$$



Therefore:

$$A = 2\pi \int_0^{2\pi} a(1 - \cos \theta) \sqrt{[a(1 - \cos \theta)]^2 + (a \sin \theta)^2} d\theta$$

$$A = 2\pi a^2 \int_0^{2\pi} (1 - \cos \theta) \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta} d\theta$$

$$A = 2\pi a^2 \int_0^{2\pi} (1 - \cos \theta) \sqrt{1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta} d\theta$$

$$A = 2\pi a^2 \int_0^{2\pi} (1 - \cos \theta) \sqrt{2} \cdot \sqrt{1 - \cos \theta} d\theta$$

$$A = 2\sqrt{2}\pi a^2 \int_0^{2\pi} (1 - \cos \theta)^{3/2} d\theta$$

$$A = \frac{64}{3} \pi a^2$$

20. Find the surface area generated by revolving about y-axis that part of the curve $6xy = x^4 + 3$, from the minimum point to $x = 2$, about the y-axis.

Solution:

$$dA = 2\pi y ds$$

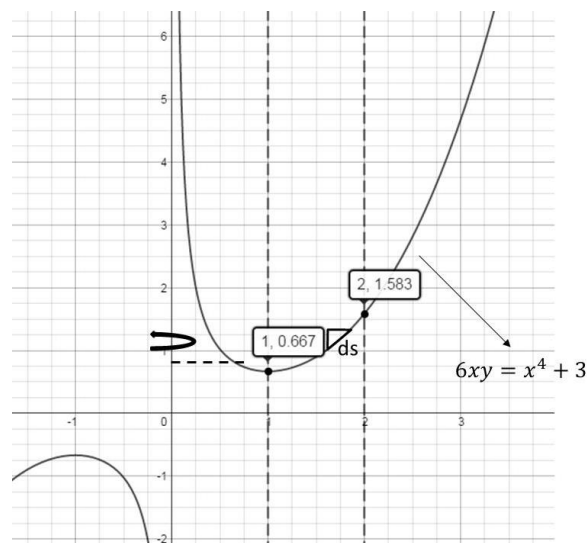
$$A = 2\pi \int y ds$$

$$A = 2\pi \int x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$6xy = x^4 + 3$$

$$y = \frac{x^3}{6} + \frac{1}{2x}$$

$$\frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2}$$



For the minimum point, set $y' = 0$:

$$\begin{aligned} \frac{x^2}{2} - \frac{1}{2x^2} &= 0 \\ x^4 - 1 &= 0 \\ x &= \pm 1 \end{aligned}$$

Therefore:

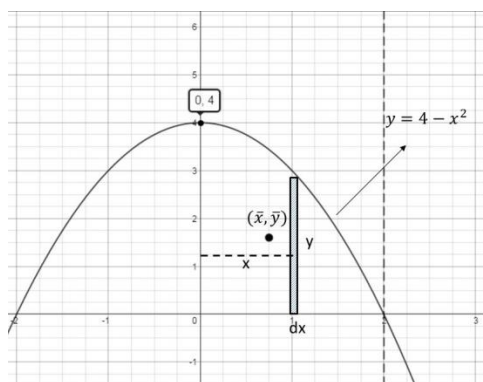
$$\begin{aligned} A &= 2\pi \int_1^2 x \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2} dx \\ A &= 13.959 \end{aligned}$$

Centroid

21. Find the centroid of the area in the first quadrant under the curve $y = 4 - x^2$.

Solution:

Referring to the figure:



$$A = \int_0^2 (4 - x^2) dx = \frac{16}{3}$$

For the x-coordinate:

$$\begin{aligned} A\bar{x} &= \int x dA \\ A\bar{x} &= \int x(y dx) \\ \frac{16}{3}\bar{x} &= \int_0^2 x(4 - x^2) dx \\ \bar{x} &= \frac{\int_0^2 x(4 - x^2) dx}{16/3} \\ \bar{x} &= \frac{3}{4} \end{aligned}$$

For the y-coordinate:

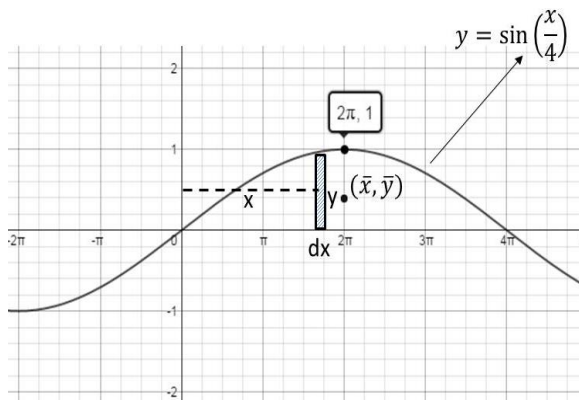
$$\begin{aligned} A\bar{y} &= \int y dA \\ A\bar{y} &= \int \frac{y}{2} (y dx) \\ \frac{16}{3}\bar{y} &= \int_0^2 \frac{(4 - x^2)^2}{2} dx \\ \bar{y} &= \frac{\int_0^2 \frac{(4 - x^2)^2}{2} dx}{16/3} \\ \bar{y} &= \frac{8}{5} \end{aligned}$$

Therefore, the centroid is $\left(\frac{3}{4}, \frac{8}{5}\right)$.

22. Find the centroid of the area in the first quadrant under the arch of $y = \sin\left(\frac{1}{4}x\right)$ nearest the x-axis.

Solution:

Referring to the figure:



$$A = \int_0^{4\pi} \left(\sin\left(\frac{1}{4}x\right)\right) dx = 8$$

For the x-coordinate:

$$A\bar{x} = \int x dA$$

$$A\bar{x} = \int x(y dx)$$

$$8\bar{x} = \int_0^{4\pi} x \left(\sin\left(\frac{1}{4}x\right)\right) dx$$

$$\bar{x} = \frac{\int_0^{4\pi} x \left(\sin\left(\frac{1}{4}x\right)\right) dx}{8}$$

$$\bar{x} = 2\pi$$

For the y-coordinate:

$$A\bar{y} = \int y dA$$

$$A\bar{y} = \int \frac{y}{2} (y dx)$$

$$8\bar{y} = \int_0^{4\pi} \frac{\left(\sin\left(\frac{1}{4}x\right)\right)^2}{2} dx$$

$$\bar{y} = \frac{\frac{1}{2} \int_0^{4\pi} \left(\sin\left(\frac{1}{4}x\right)\right)^2 dx}{8}$$

$$\bar{y} = \frac{\pi}{8}$$

Therefore, the centroid is $\left(2\pi, \frac{\pi}{8}\right)$.

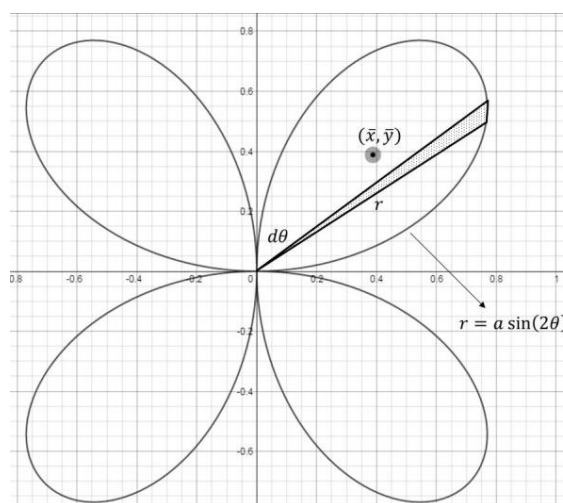
23. Find the centroid of the first quadrant area bounded by the curve $r = a \sin 2\theta$.

Solution:

r	0	$\frac{\sqrt{3}}{2}a$	a	$\frac{\sqrt{3}}{2}a$	0
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

Plotting the points:

Referring to the figure:



$$A = \frac{1}{2} \int_0^{\frac{\pi}{2}} (a \sin 2\theta)^2 d\theta = \frac{\pi}{8}$$

For the x-coordinate:

$$\begin{aligned} A\bar{x} &= \int x dA \\ \frac{\pi}{8}\bar{x} &= \int \left(\frac{2}{3}r \cos \theta\right) \cdot \frac{1}{2}r^2 d\theta \\ \frac{\pi}{8}\bar{x} &= \frac{1}{3} \int_0^{\frac{\pi}{2}} (a \sin 2\theta)^3 \cos \theta d\theta \\ \bar{x} &= \frac{\frac{1}{3} \int_0^{\frac{\pi}{2}} (a \sin 2\theta)^3 \cos \theta d\theta}{\pi/8} \\ x &= \frac{128a}{105\pi} \approx 0.388a \end{aligned}$$

For the y-coordinate:

$$\begin{aligned} A\bar{y} &= \int y dA \\ \frac{\pi}{8}\bar{y} &= \int \left(\frac{2}{3}r \sin \theta\right) \cdot \frac{1}{2}r^2 d\theta \\ \frac{\pi}{8}\bar{y} &= \frac{1}{3} \int_0^{\frac{\pi}{2}} (a \sin 2\theta)^3 \sin \theta d\theta \\ \bar{y} &= \frac{\frac{1}{3} \int_0^{\frac{\pi}{2}} (a \sin 2\theta)^3 \sin \theta d\theta}{\pi/8} \\ y &= \frac{128a}{105\pi} \approx 0.388a \end{aligned}$$

Therefore, the centroid is $\left(\frac{128a}{105\pi}, \frac{128a}{105\pi}\right)$.

24. Find the x-coordinate of the centroid of the volume generated by revolving about x-axis the area in the second quadrant under the curve $y = e^x$.

Solution:

Referring to the figure:

$$V = \pi \int y^2 dx = \pi \int_{-\infty}^0 (e^x)^2 dx = \frac{\pi}{2}$$

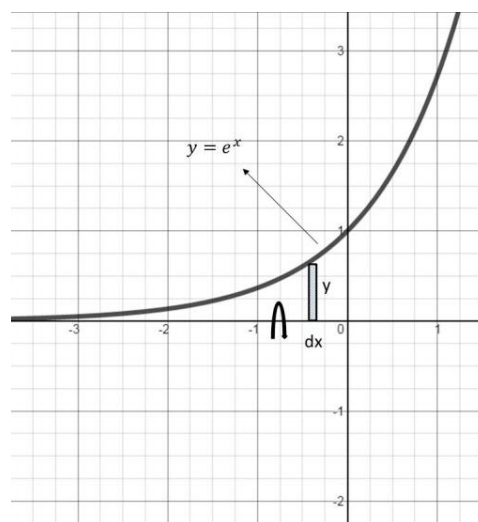
$$\frac{\pi}{2} \bar{x} = \int \pi xy^2 dx$$

$$\frac{\pi}{2} \bar{x} = \pi \int xy^2 dx$$

$$\frac{1}{2} \bar{x} = \int xy^2 dx$$

$$\frac{1}{2} \bar{x} = \int_{-\infty}^0 x(e^x)^2 dx$$

$$\frac{1}{2} \bar{x} = \lim_{b \rightarrow -\infty} \int_b^0 xe^{2x} dx$$



Considering the integral on the right side of the equation:

Using integration by parts:

$$\int xe^{2x} dx$$

Let:

$$u = x; du = dx$$

$$dv = e^{2x} dx; v = \frac{1}{2} e^{2x}$$

We have:

$$\int xe^{2x} dx = \frac{1}{2} xe^{2x} - \frac{1}{2} \int e^{2x} dx$$

$$\int xe^{2x} dx = \frac{1}{2} xe^{2x} - \frac{1}{4} e^{2x}$$

Hence,

$$\frac{1}{2} \bar{x} = \lim_{b \rightarrow -\infty} \left[\frac{1}{2} xe^{2x} - \frac{1}{4} e^{2x} \right]_b^0$$

$$\frac{1}{2} \bar{x} = \lim_{b \rightarrow -\infty} \left[\left(0 - \frac{1}{4} \right) - \left(\frac{1}{2} be^{2b} - \frac{1}{4} e^{2b} \right) \right]$$

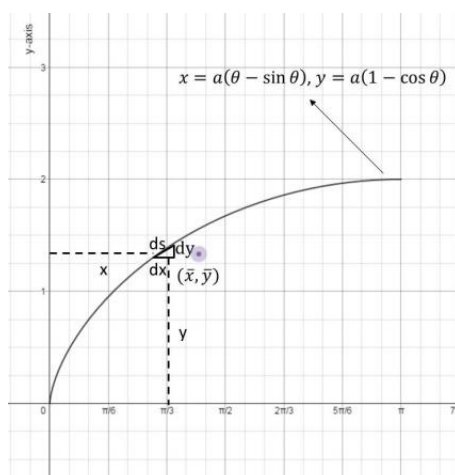
$$\frac{1}{2} \bar{x} = -\frac{1}{4} - 0$$

$$\bar{x} = -\frac{1}{2}$$

25. Find the centroid of the arc from cusp to vertex of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.

Solution:

Referring to the figure:



We have:

$$s \cdot \bar{x} = \int x \, ds$$

$$\int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \cdot \bar{x} = \int x \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

$$x = a(\theta - \sin \theta); \frac{dx}{d\theta} = a(1 - \cos \theta)$$

$$y = a(1 - \cos \theta); \frac{dy}{d\theta} = a(\sin \theta)$$

$$s = \int_0^\pi \sqrt{[a(1 - \cos \theta)]^2 + a^2 \sin^2 \theta} d\theta$$

$$s = 4$$

For the x-coordinate:

Substituting:

$$4\bar{x} = \int_0^\pi a(\theta - \sin \theta) \sqrt{[a(1 - \cos \theta)]^2 + a^2 \sin^2 \theta} d\theta$$

$$4\bar{x} = a^2 \int_0^\pi (\theta - \sin \theta) \sqrt{[(1 - \cos \theta)]^2 + \sin^2 \theta} d\theta$$

$$\bar{x} = \frac{a^2 \int_0^\pi (\theta - \sin \theta) \sqrt{[(1 - \cos \theta)]^2 + \sin^2 \theta} d\theta}{4}$$

$$\bar{x} = \frac{4}{3} a$$

For the y-coordinate:

$$4\bar{y} = \int_0^\pi a(1 - \cos \theta) \sqrt{[a(1 - \cos \theta)]^2 + a^2 \sin^2 \theta} d\theta$$

$$4\bar{y} = a^2 \int_0^\pi (1 - \cos \theta) \sqrt{[(1 - \cos \theta)]^2 + \sin^2 \theta} d\theta$$

$$\bar{y} = \frac{a^2 \int_0^\pi (1 - \cos \theta) \sqrt{[(1 - \cos \theta)]^2 + \sin^2 \theta} d\theta}{4}$$

$$\bar{y} = \frac{4}{3} a$$

Therefore, the centroid is $\left(\frac{4}{3} a, \frac{4}{3} a\right)$.

Moment of Inertia

26. Find the moment of inertia of the area of the loop $y^2 = x^2(1 - x)$, with respect to y-axis.

Solution:

Referring to the figure:

We have:

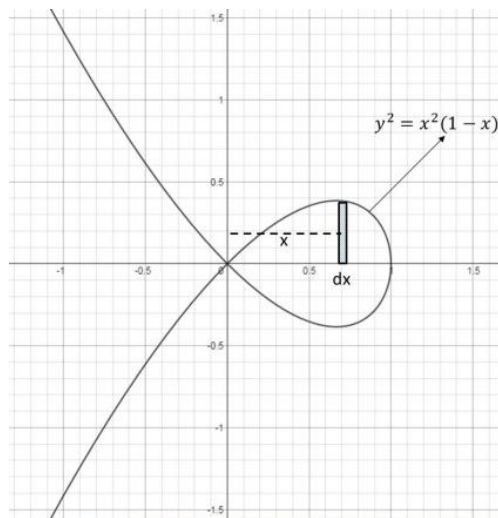
$$I_y = \int x^2 y \, dx$$

By symmetry:

$$I_y = 2 \int_0^1 x^2 (x\sqrt{1-x}) \, dx$$

$$I_y = 2 \int_0^1 x^3 \sqrt{1-x} \, dx$$

$$I_y = \frac{64}{315} \text{ units}^4$$



27. Find the moment of inertia of the area under the curve $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$, with respect to the y-axis.

Solution:

Referring to the figure:

We have:

$$I_y = \int x^2 y \, dx$$

$$I_y = \int_0^{\pi/2} x^2 \sin x \, dx$$

Using integration by parts:

$$\int x^2 \sin x \, dx$$

Let:

$$u = x^2; \, du = 2x \, dx$$

$$dv = \sin x \, dx; \, v = -\cos x$$

We have:

$$\int u \, dv = uv - \int v \, du$$

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2 \int x \cos x \, dx$$

Again, using integration by parts, let:

$$u = x; \, du = dx$$

$$dv = \cos x \, dx; \, v = \sin x$$

Again, we have:

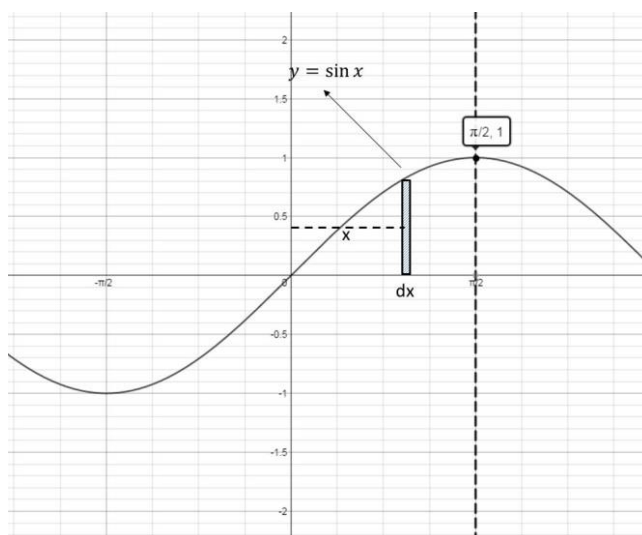
$$\int x^2 \sin x \, dx = -x^2 \cos x + 2 \left[x \sin x - \int \sin x \, dx \right]$$

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x$$

Therefore,

$$I_y = \int_0^{\pi/2} x^2 \sin x \, dx = \left[-x^2 \cos x + 2x \sin x + 2 \cos x \right]_0^{\pi/2}$$

$$I_y = \pi - 2 \approx 1.1416 \text{ units}^4$$



28. What is the moment of inertia of the area bounded by the curve $2x^2 + 2x - y - 2 = 0$ and the line $y = 2x$, with respect to the line $x = 1$?

Solution:

Solving for the roots of the quadratic equation:

$$y = 2x^2 + 2x - 2$$

Using MODE 5-3:

$$x_1 = \frac{-1 + \sqrt{5}}{2} = 0.618$$

$$x_2 = \frac{-1 - \sqrt{5}}{2} = -1.618$$

Now, solving for the point(s) of intersection:

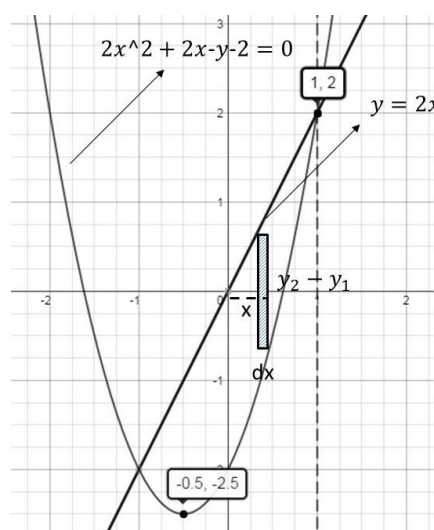
$$2x^2 + 2x - 2 = 2x$$

$$2x^2 - 2 = 0$$

$$2(x^2 - 1) = 0$$

$$x = \pm 1$$

Now, referring to the figure:



$$I_{x=1} = \int (1 - x)^2 (y_2 - y_1) dx$$

$$I_{x=1} = \int_{-1}^1 (1 - x)^2 [2x - (2x^2 + 2x - 2)] dx$$

$$I_{x=1} = \int_{-1}^1 (1 - x)^2 (2 - 2x^2) dx$$

$$I_{x=1} = \frac{16}{5} = \mathbf{3.2 \text{ units}^4}$$

STATISTICS

SITUATION 1

Given: 12 34 45 23 87 91 121

1. Find the mean.

A. 54	C. 59
B. 57	D. 61

2. Find the population standard deviation.

A. 35.56	C. 33.45
B. 37.75	D. 38.21

3. Find the sample standard deviation.

A. 40.78	C. 41.22
B. 43.34	D. 44.34

4. How many 4 digit numbers can be formed using digits 1, 2, 3, 4, 6 and 7 if the digits are distinct?

A. 360	C. 400
B. 320	D. 420

SITUATION 2

The specification for a job calls for a class “B” mix with a minimum compressive strength of 3,000 psi at 28 days. The result of 125 compressive tests are tabulated in the following table.

28 days compressive strength (psi)	Number of Tests
2,800	2
2,900	4
3,000	6
3,100	11
3,200	24
3,300	37
3,400	19
3,500	12
3,600	6
3,700	4

5. The mean strength is:

A. 3311.25	C. 3414.34
B. 3289.6	D. 3411.22

6. The standard deviation is:

A. 185.33	C. 183.23
B. 195.22	D. 189.45

41. Ivan conducted a survey for 500 tele-viewers on what type of television shows they watch in TV. The survey results are as follows:

115 watch soap opera	70 watch documentaries and live games
285 watch live games	50 watch documentaries and soap opera
195 watch documentaries	45 watch live games and soap opera
50 watch other type of shows	

How many watch documentaries only?

- A. 75 C. 85
B. 95 D. 105

STATISTICS SOLUTIONS

SITUATION 1

Go to **MODE 3[STAT] 1[1-VAR]**

Input the given data

Press **[AC]**

1. To get the mean, \bar{x} , the syntax is **SHIFT 1[STAT] 4[Var] 2[\bar{x}]**
 $\bar{x} = 59$

2. To get the population standard deviation, σ_x , the syntax is **SHIFT 1[STAT] 4[Var] 3[σ_x]**
 $\sigma_x = 37.75$

3. To get the sample standard deviation, s_x , the syntax is **SHIFT 1[STAT] 4[Var] 4[s_x]**
 $s_x = 40.78$

4. To have four digit numbers with distinct digits, 4 numbers must be taken from the given 6 digits by order:
 $nPr = 6P4 = 360$

SITUATION 2

Go to **MODE 3[STAT] 1[1-Var]**

Turn on the frequency thru **SHIFT MODE ▼ [Down Key] 4[STAT] 1[On]**

Write down the data

x	FREQ
2800	2
2900	4
3000	6
3100	11
3200	24
3300	37
3400	19
3500	12
3600	6
3700	4

Press **[AC]**

5. To get the mean, \bar{x} , the syntax is **SHIFT 1[STAT] 4[Var] 2[\bar{x}]**
 $\bar{x} = 3289.6$

6. To get the population standard deviation, σ_x , the syntax is **SHIFT 1[STAT] 4[Var] 3[σ_x]**
 $\sigma_x = 183.23$

SITUATION 2

Go to **MODE 3[STAT] 1[1-VAR]**

Turn on the frequency thru **SHIFT MODE ▼ [Down Key] 4[STAT] 1[On]**

Write down the data

x	FREQ
30	1
42	2
50	4
60	6

Press **[AC]**

7. To get the mean, \bar{x} , the syntax is **SHIFT 1[STAT] 4[Var] 2[\bar{x}]**

$$\bar{x} = 51.85$$

8. To get the population standard deviation, σ_x , the syntax is **SHIFT 1[STAT] 4[Var] 3[σ_x]**

$$\sigma_x = 9.13$$

9. Go to **MODE 3[STAT] 1[1-Var]**

Turn on the frequency thru **SHIFT MODE ▼ [Down Key] 4[Stat] 1[On]**

Write down the data

x	FREQ
30	1
42	2
50	4
60	6

Press **[AC]**

To get the population standard deviation, σ_x , the syntax is **SHIFT 1[STAT] 4[Var] 3[σ_x]**

$$\sigma_x = 12.67$$

10. Code requires an ARRANGEMENT; therefore, permutation is the solution.

There are 4 possible cases.

There is only 1 flag to be used: $4P1 = 4$

There are 2 flags to be used: $4P2 = 12$

There are 3 flags to be used: $4P3 = 24$

There are 4 flags to be used: $4P4 = 24$

$$4P1 + 4P2 + 4P3 + 4P4 = 64$$

11. To get a score in bowling, the player must hit 1 pin, 2 pins, 3 pins up to 10 pins in no particular order (hence, combination is the solution)

$$\sum_{x=1}^{10} 10C_x = 10C_1 + 10C_2 + \dots + 10C_{10} = 1,023$$

12. MATHEMATICS: $M = 2, A = 2, T = 2, H = 1, E = 1, I = 1, C = 1, S = 1$

$$\frac{11!}{2! \times 2! \times 2! \times 1! \times 1! \times 1! \times 1! \times 1!} = 4,989,600$$

13. MEGAREVIEW: $M = 1, E = 3, G = 1, A = 1, R = 1, V = 1, I = 1, W = 1$

$$\frac{10!}{3!} = 604,800$$

14. To get the number arrangement of the trees in the circular lot:

$$(n - 1)! = (10 - 1)! = 362,880$$

SITUATION 4

15. There are two cases possible:

Case 1: Boys sit on the first 3 seats; then, followed by the 3 girls

$BBBGGG = (3!) (3!) = 36$

Case 2: Girls sit on the first 3 seats; then, followed by the 3 boys

$GGGBBB = (3!) (3!) = 36$

Combine the two case, $36 + 36 = 72$

16. There are four cases possible:

GGGBBB, BGGGBB, BBGGGB, BBBGGG

The solution is $3! \times 4! = 144$. $3!$ because you can arrange the 3 girls within themselves, $4!$ because you can arrange the 3 boys and girls (3 girls act as one unit).

17. There are four cases possible:

AAAAMMM, MAAAAMM, MMAAAAAM, MMMAAAA

Possible ways the armies are together:

$4 \times 4! \times 3! = 576$

18. You can enter one of the 9 doors, but you can exit on 8 other doors that you did not enter.

$9 \times 8 = 72$

19. $5C1 + 5C2 + 5C3 + 5C4 + 5C5 = 31$

20. $C(n,2) = 91$ Use trial and error for the 4 choices, $n = 14$.

21. $8 \times 2 \times 10 = 160$

22. For Cities A and B: possible ways = $5P2 = 20$; for B and C, possible ways = $4P2 = 12$

Possible ways = $20 \times 12 = 240$

23. $6C5 \times 6C3 = 120$. (Note he must answer the 1st 5, so only 6 questions remain, and he must answer only 3)

24. $9C7 \times 6C5 = 216$

25. $4^{50} = 1.27 \times 10^{30}$

26. $\frac{8!}{2! \times 2! \times 2! \times 2!} = 2,520$

$$27. \frac{12!}{2! \times 4! \times 6!} = 13,860$$

$$28. 10C1 \times 8C1 \times 6C1 = 480$$

$$29. 6C3 = 20$$

$$30. 15P5 = 360,360$$

SITUATION 5

$$31. 10C3 = 120$$

$$32. 9C2 = 36$$

33. The four regional delegates can be arranging in a circle in $3!$ ways. The number of ways they can sit together = $3! \times (3! \times 4! \times 4! \times 2!) = 41,472$.

$$34. \frac{12!}{4! \times 4! \times 4!} = 34,650$$

35. We can use only 1, 3, 4 in the hundreds' digit.
 $3 \times 4 \times 3 = 36$

SITUATION 6

For mean, median and mode problems, arrange the given data in increasing (or decreasing order).

Go to **MODE 3[STAT] 1[1-VAR]**

Turn on the frequency thru **SHIFT MODE ▼ [Down Key] 4[STAT] 1[On]**

Write down the data

x	FREQ
53	1
55	1
56	2
58	2
59	2
60	1
61	2
62	3
64	1
65	2
67	1
68	2
70	1

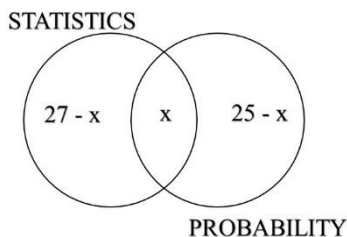
Press **[AC]**

36. To get the mean, \bar{x} , the syntax is **SHIFT 1[STAT] 4[Var] 2[\bar{x}]**
 $\bar{x} = 61.38$

37. Say the total number of data is denoted as “n” which is equal to 21 in this problem, then the value indicated in “(n + 1)/2” from lowest (or from highest) is the median. The median is **61**.

38. The mode of a set of data is the one with the highest frequency in the set. The mode is **62**.

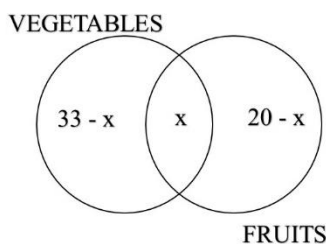
39.



$$(27 - x) + x + (25 - x) = 40$$

$$x = 12$$

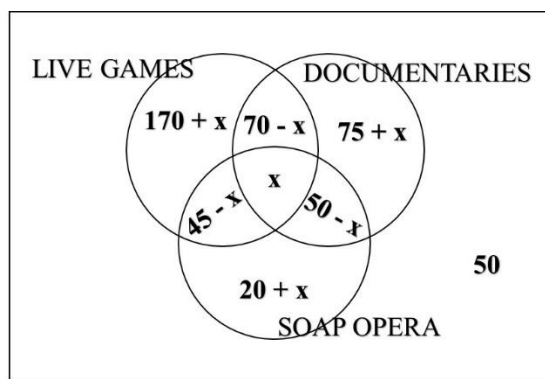
40.



$$(33 - x) + x + (20 - x) = 40$$

$$x = 13$$

41.



For live games tele viewers alone:

$$285 - (70 - x) - x - (45 - x) = 170 + x$$

For documentaries tele viewers alone:

$$195 - (70 - x) - x - (50 - x) = 75 + x$$

For soap opera tele viewers alone:

$$115 - (45 - x) - x - (50 - x) = 20 + x$$

For the tele-viewers of live games, documentaries and soap operas:

$$285 + (75 + x) + (50 - x) + (20 + x) + 50 = 500; x = 20$$

Documentaries tele-viewer only:

$$75 + x = 75 + 20 = 95$$

Notes:

For nPr, ENTER n SHIFT × r

For nCr, ENTER n SHIFT ÷ r

Where n and r are whole numbers given that $n \geq r$

PROBABILITY

1. The lotto uses numbers 1 to 42. A winning number uses 6 different numbers in any order. What is your chance of winning if you bet one ticket?

A. $1/4534568$	C. $1/6580668$
B. $1/5245786$	D. $1/2341668$

2. If you roll a pair of dice one time, what is the probability of getting a sum of 9?

A. $1/9$	C. $1/4$
B. $1/6$	D. $1/3$

3. A point is selected inside a circle. Find the probability that the point is closer to the center of the circle.

A. $1/3$	C. $1/2$
B. $1/4$	D. $1/5$

4. A die is constructed so that a 1 or a 2 occurs twice as a 3, 4 or a 6 and a 5 occurs thrice as a 1 or a 2. If the die is tossed once. Find the probability that an even number will occur?

A. $4/9$	C. $2/9$
B. $1/9$	D. $4/13$

5. If the odds against event E are 2:7, find the probability of success.

A. $2/9$	C. $7/9$
B. $9/14$	D. $1/2$

6. In a survey concerning the use of three different brand of calculator, it was found that 55% use calculator A, 50% use calculator B, 40% use calculator C, 30% use calculator A and B, 20% use calculator A and C, 12% use calculator B and C and only 10% use all the three calculators. What percentage of the population do not use calculators?

A. 7%	C. 6%
B. 4%	D. 3%

SITUATION 1. Three machines A, B and C produce respectively 50%, 30% and 20% of the total number of item of a factory. The percentages of defective output of these machines are 3%, 4% and 5%.

7. If an item is selected at random, find the probability that the item is defective.

A. 0.037	C. 0.012
B. 0.045	D. 0.047

8. Suppose an item is selected at random and found to be defective, find the probability that the item was produced by machine A.

A. $15/37$	C. $12/37$
B. $11/37$	D. $18/37$

9. Of 120 engineers surveyed, 60 are working in the academe, 50 are working in the industry, and 20 are working both in academe and in industry. If an engineer is chosen at random, find the probability that the engineer is working either in academe or in industry.

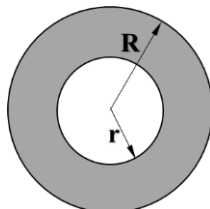
A. $1/4$	C. $1/2$
B. $3/4$	D. $4/5$

10. A pair of fair dice is tossed. Find the probability that the maximum of the two numbers is greater than 4.
 A. $\frac{4}{9}$ C. $\frac{5}{9}$
 B. $\frac{2}{9}$ D. $\frac{3}{9}$
11. Three boys and three girls sit in a row. Find the probability that the 3 girls sit together.
 A. $\frac{1}{5}$ C. $\frac{2}{5}$
 B. $\frac{3}{5}$ D. $\frac{2}{3}$
12. A fair coin is tossed 10 times. Compute the probability of getting at least 7 heads.
 A. $\frac{9}{64}$ C. $\frac{11}{64}$
 B. $\frac{7}{64}$ D. $\frac{13}{64}$
13. The probability of getting a credit in an examination is $\frac{1}{3}$. If three students are selected at random, what is the probability that at least one of them get the credit?
 A. $\frac{2}{3}$ C. $\frac{19}{27}$
 B. $\frac{20}{27}$ D. $\frac{7}{9}$
14. In a basketball game, the free throw average is 0.65. Find the probability that a player misses one shot of the three free throws?
 A. 0.441 C. 0.422
 B. 0.444 D. 0.451
15. What is the probability of getting at least 3 heads when a fair coin is tossed 6 times.
 A. $\frac{19}{32}$ C. $\frac{17}{32}$
 B. $\frac{21}{32}$ D. $\frac{15}{32}$
16. Find the probability of getting a prime number thrice by tossing a fair die 5 times.
 A. 0.4225 C. 0.3125
 B. 0.375 D. 0.1626
17. A fair coin is tossed 10 times. What is the probability of getting 4 tails and 6 heads?
 A. $\frac{105}{512}$ C. $\frac{13}{64}$
 B. $\frac{51}{256}$ D. $\frac{25}{128}$
18. What is the probability of a family with 5 children of having 3 boys and 2 girls? Assume they have equal probability.
 A. $\frac{5}{16}$ C. $\frac{1}{4}$
 B. $\frac{3}{16}$ D. $\frac{3}{8}$
19. Find the probability that a person flipping a coin gets the 3rd head on the 7th flip.
 A. 0.1144 C. 0.1172
 B. 0.1221 D. 0.1344
20. Find the probability that a person tossing a fair coin gets the 1st tail on the 4th flip.
 A. 0.035 C. 0.0012
 B. 0.0625 D. 0.1122

39. A bag contains 3 white balls and 5 red balls. If two balls are drawn at random in succession without returning the first ball drawn, what is the probability that the balls drawn are both red?
 A. $\frac{3}{14}$ C. $\frac{1}{3}$
 B. $\frac{5}{14}$ D. $\frac{3}{7}$
40. A bag contains 3 white and 5 black balls. If two balls are drawn in succession without replacement, what is the probability that both balls are white?
 A. $\frac{1}{14}$ C. $\frac{3}{28}$
 B. $\frac{4}{7}$ D. $\frac{5}{28}$
41. An urn contains 4 black balls and 6 white balls. What is the probability of getting 1 black and 1 white ball in two consecutive draws from the urn?
 A. $\frac{8}{15}$ C. $\frac{7}{15}$
 B. $\frac{6}{15}$ D. $\frac{4}{15}$
42. Three light bulbs are chosen at random from 15 bulbs of which 5 are defective. Find the probability that none is defective.
 A. $\frac{23}{91}$ C. $\frac{24}{91}$
 B. $\frac{20}{91}$ D. $\frac{33}{91}$
43. A pack of cards contains 52 cards: 13 spades, 13 clubs, 13 hearts and 13 diamonds. Of the 52 cards, 4 are aces one from each suit. The hearts and the diamonds are colored red, the spades and clubs are black. Four cards are drawn from the pack, each card being returned to the pack before the next card is drawn. Find the probability that all are clubs.
 A. $\frac{1}{4}$ C. $\frac{1}{100}$
 B. $\frac{1}{256}$ D. $\frac{1}{512}$
44. Refer to problem 43. If 5 cards are drawn simultaneously, find the probability to get all the 4 aces.
 A. $\frac{1}{54145}$ C. $\frac{2}{54145}$
 B. $\frac{4}{54145}$ D. $\frac{9}{54145}$
45. A fair die is tossed 8 times. What is the probability of obtaining the faces 5 and 6 twice and each of the other once.
 A. 0.004 C. 0.005
 B. 0.006 D. 0.007
46. The painted light bulbs produced by a company are 50% red, 30% blue and 20% green. In a sample of 5 bulbs, find the probability that 2 are red, 1 is green and 2 are blue.
 A. 0.07 C. 0.08
 B. 0.09 D. 0.10

PROBABILITY SOLUTIONS

1. $1/42C6 = 1/5,245,786$
2. The possible sums (2, 7), (7, 2), (4, 5), (5, 4)
Probability = $4 / (6 \times 6) = 4/36 = 1/9$
3. Consider $R = 2r$ to assure the points within the bigger circle can be differentiated as near or away the center of the circle.



$$\begin{aligned} \text{Probability of being closer to the center} &= \frac{A_{\text{smaller circle}}}{A_{\text{bigger circle}}} \\ &= \frac{\pi r^2}{\pi (2r)^2} = \frac{1}{4} \end{aligned}$$

4. Let x = probability of getting either 3, 4 or 6.
Then, probability of getting either 1 or 2 is $2x$.
Furthermore, probability of getting 5 is $3(2x) = 6x$.

$$\begin{aligned} P(1) + P(2) + P(3) + P(4) + P(5) + P(6) &= 1 \\ 2x + 2x + x + x + 6x + x &= 1 \\ x &= \frac{1}{13} \end{aligned}$$

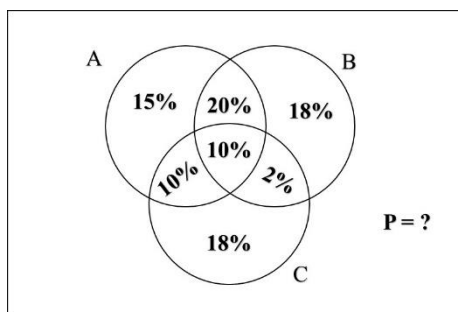
Hence,

$$P(1) = \frac{2}{13}, P(2) = \frac{2}{13}, P(3) = \frac{1}{13}, P(4) = \frac{1}{13}, P(5) = \frac{6}{13}, P(6) = \frac{1}{13}$$

The even numbers are 2, 4, and 6. Then, the probability of getting an even number is:

$$P(\text{even}) = P(2) + P(4) + P(6) = \frac{2}{13} + \frac{1}{13} + \frac{1}{13} = \frac{4}{13}$$

5. $5 / (2 + 7) = 7/9$
6. The sum of users of calculator = $15\% + 20\% + 10\% + 10\% + 18\% + 2\% + 18\% = 93\%$



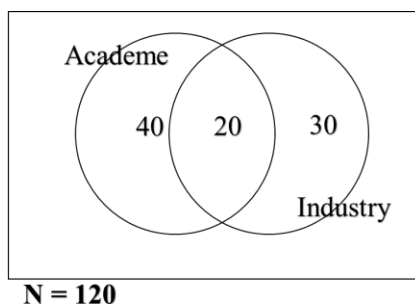
% that do not use calculator = $100 - 93 = 7\%$

SITUATION 1

7. Probability that item is defective = $50\% (3\%) + 30\% (4\%) + 20\%(5\%) = 0.037$

8. $P(\text{produced by machine A and defective}) = 50\% \times 3\% = 0.015$
 $P(\text{produced by machine A given that it is defective}) = 0.015/0.037 = 15/37$

9. Probability of Engineers Working in Academe or in Industry = $(40 + 20 + 30)/ 120 = 3/4$



10. There are 2 cases: (a) The first number should be 5 or 6 and the last number could be any number; and (b) The last number should be 5 or 6 and the first number could be any number.

Here are the possible pairs:

- | | | | |
|-----|-----|-----|-----|
| 5-1 | 1-5 | 6-1 | 1-6 |
| 5-2 | 2-5 | 6-2 | 2-6 |
| 5-3 | 3-5 | 6-3 | 3-6 |
| 5-4 | 4-5 | 6-4 | 4-6 |
| 5-5 | 6-5 | 5-6 | |
| | 6-6 | | |

Probability (maximum of two numbers greater than 4) = $20/36 = 5/9$

11. Probability that the 3 girls sit together = Number of ways the 3 girls sit together/ Number of ways the 6 people sit in a row

Number of ways the 3 girls sit together = $4! \times 3! = 144$

Number of ways the 6 people sit in a row = $6! = 720$

Then, probability that the 3 girls sit together = $144/720 = 1/5$

12. Let $p = P(H)$ and $q = P(T)$. Since the coin is fair, $p=q=0.50$

$$P(X = r) = nCr \cdot p^r \cdot q^{n-r}$$

$$P(X \geq 7) = P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$$

$$P(X \geq 7) = \sum_{x=7}^{10} (nCr \cdot p^r \cdot q^{n-r}) = \sum_{r=7}^{10} 10Cr(0.50)^r(0.50)^{10-r}$$

$$P(X \geq 7) = \frac{11}{64}$$

13. Let $p = P(\text{Credit})$ and $q = P(\text{No credit})$

$$P(X = r) = nCr \cdot p^r \cdot q^{n-r}$$

$$P(X \geq 1) = P(X = 1) + P(X = 2) + P(X = 3)$$

$$P(X \geq 1) = \sum_{x=1}^3 (nCr \cdot p^r \cdot q^{n-r}) = \sum_{r=1}^3 3Cr \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{3-r}$$

$$P(X \geq 1) = \frac{19}{27}$$

14. Let $p = \text{probability of shooting} = 0.65$; $q = \text{probability of missing a shot} = 0.35$

When a player misses one shot, the other two should be a made basket.

$$P(X = r) = nCr \cdot p^r \cdot q^{n-r}$$

$$P(X = 2) = 3C2(0.65)^2(0.35)^{3-2}$$

$$P(X = 2) = \mathbf{0.4436}$$

15. Let $p = P(\text{Head})$ and $q = P(\text{Tail})$. Since the coin is fair, $p=q=0.50$

$$P(X = r) = nCr \cdot p^r \cdot q^{n-r}$$

$$P(X \geq 3) = P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)$$

$$P(X \geq 3) = \sum_{x=3}^6 (nCr \cdot p^r \cdot q^{n-r}) = \sum_{r=3}^6 6Cr(0.50)^r(0.50)^{6-r}$$

$$P(X \geq 3) = \frac{21}{32}$$

16. Let $p = \text{probability of getting prime number} = 0.50$, $q = \text{probability of getting not a prime number} = 0.50$

$$P(X = r) = nCr \cdot p^r \cdot q^{n-r}$$

$$P(X = 3) = 5C3(0.50)^3(0.50)^{5-3}$$

$$P(X = 3) = \frac{5}{16} = \mathbf{0.3125}$$

17. Let $p = \text{probability of getting head} = 0.50$, $q = \text{probability of getting tail} = 0.50$

$$P(X = 4) = nCr \cdot p^r \cdot q^{n-r}$$

$$P(X = 4) = 10C4(0.50)^4(0.50)^{10-4}$$

$$P(X = 4) = \frac{105}{512}$$

18. Let $p = \text{probability of having a boy} = 0.50$, $q = \text{probability of having a girl} = 0.50$

$$P(X = r) = nCr \cdot p^r \cdot q^{n-r}$$

$$P(X = 3) = 5C3(0.50)^3(0.50)^{5-3}$$

$$P(X = 3) = \frac{5}{16} = \mathbf{0.3125}$$

19. Let $p = \text{probability of getting head} = 0.50$, $q = \text{probability of getting tail} = 0.50$

In this problem, the 3rd head is assured to be located at the 7th flip. The first 2 heads are randomly occurred on the first 6 flips.

$$P(\text{3rd head on the 7th flip}) = [6C2(0.50)^2(0.50)^{6-2}](0.50) = \mathbf{0.1171875}$$

20. Let $p = \text{probability of getting head} = 0.50$, $q = \text{probability of getting tail} = 0.50$

In this problem, the 1st tail is assured to be located at the 4th flip. The first 3 toss gives all heads.

$$P(\text{1st tail on the 4th flip}) = [3C3(0.50)^3(0.50)^{3-3}](0.50) = \mathbf{0.0625}$$

21. There are 3 Possible Outcomes:

Case	Probability
If the pilot passed the test on the first try	0.7
If the pilot failed the first try and passed on the second try	$[1-0.7] (0.7) = 0.21$
If the pilot failed the first two try and passed on the third try	$[1-0.7] [1-0.7] (0.7) = 0.063$

Probability the pilot passed the written test before the 4th try = $0.7 + 0.21 + 0.063 = 0.973$

22.

Event	Probability, p	Consequence, x
Getting 1 or 6	2/6	20
Getting other numbers	4/6	-10

$$\text{Expectation} = \sum p_i x_i = \frac{2}{6} (20) + \frac{4}{6} (-10) = 0$$

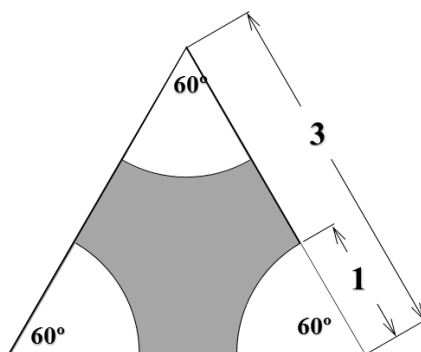
23. Expected Cost = $\sum p_i x_i = 1(0.2) + 2(0.28) + 3(0.18) + 4(0.23) + 5(0.11) = \text{PhP } 2.77$

24.

Event	Probability, p	Consequence, x
Getting 2 Heads	$P(X = 2) = 2C2(0.50)^2(0.50)^{2-2}$ $P(X = 2) = 0.25$	PhP 2
Getting 1 Head	$P(X = 1) = 2C1(0.50)^1(0.50)^{2-1}$ $P(X = 1) = 0.50$	PhP 1
Getting Tail	$P(X = 0) = 2C0(0.50)^0(0.50)^{2-0}$ $P(X = 0) = 0.25$	-PhP 5

$$\text{Expected Cost} = \sum p_i x_i = 2(0.25) + 1(0.50) - 5(0.25) = -0.25$$

25. To assure that the point is greater than 1 from any corner, the circular sector of radius 1 is constructed from the corners. The gray region is the area at which the point has a distance greater than one from any corner.



Probability that distance is greater than 1 from any corner = Area of the Gray Region / Area of the Triangle

$$\text{Area of the 3 sectors} = \frac{1}{2} (r^2 \theta) (3) = \frac{1}{2} (1^2) \left(\frac{60\pi}{180} \right) (3) = \frac{\pi}{2}$$

$$\text{Area of the Triangle} = \frac{s^2 \sqrt{3}}{4} = \frac{3^2 \sqrt{3}}{4} = \frac{9\sqrt{3}}{4}$$

$$\text{Probability (distance greater than 1 from any corner)} = \frac{\frac{9\sqrt{3}}{4} - \frac{\pi}{2}}{\frac{9\sqrt{3}}{4}} = 0.597$$

26.

Face of the Dice	Probability	Consequence
1	1/6	-1
2	1/6	2
3	1/6	3
4	1/6	-4
5	1/6	5
6	1/6	-6

NOTE: Negative sign is affixed when the face of dice is 1, 4 and 6 because they are not prime numbers. Hence, the consequence is they will lose money when these number appears.

$$\text{Expected Winnings} = 1/6(-1) + 1/6(2) + 1/6(3) + 1/6(-4) + 1/6(5) - 1/6(6) = -1/6$$

27.

Event	Number of Defects	Probability
There are no defective items obtained	0	$(3C0)(9C3)/12C3 = 84/220$
There is 1 defective item obtained	1	$(3C1)(9C2) /12C3 = 108/220$
There are 2 defective items obtained	2	$(3C2)(9C1) /12C3 = 27 /220$
There are 3 defective items obtained	3	$(3C3)(9C0) /12C3 = 1/220$

$$\text{Expected number of Defective Items} = 0(84/220) + 1(108/220) + 2(27/220) + 3(1/220) = 3/4$$

SITUATION 2

28. Apply Poisson's Distribution formula:

$$P(x) = \frac{e^{-\mu}(\mu)^x}{x!}$$

$$P(x = 3) = \frac{e^{-3}(3)^5}{5!} = \mathbf{0.10082}$$

29. Apply Poisson's Distribution formula:

$$P(X < 3) = \sum_{x=0}^2 \frac{e^{-3}(3)^x}{x!} = \mathbf{0.4232}$$

30. Apply Poisson's Distribution formula:

$$P(X \geq 4) = 1 - P(x < 4) = 1 - \sum_{x=0}^3 \frac{e^{-2}(2)^x}{x!} = \mathbf{0.1428}$$

31. $\mu = 0.02(100) = 2$

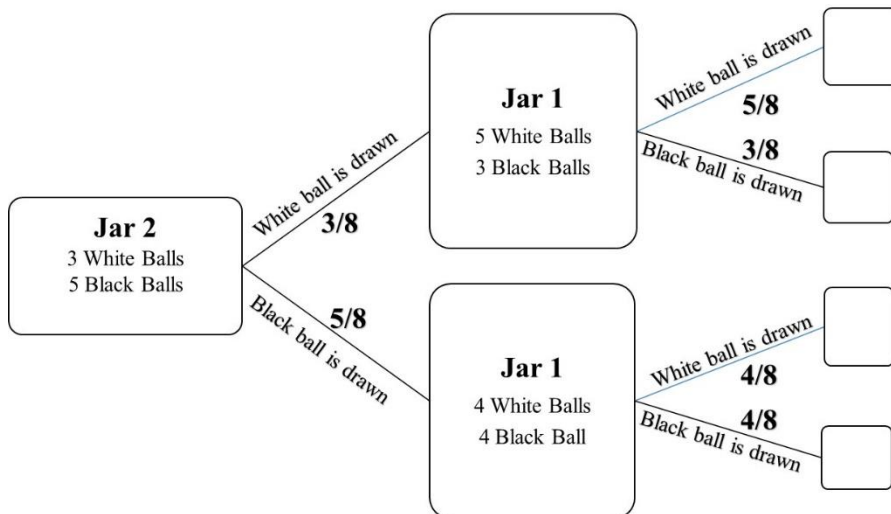
$$P(x = 3) = \frac{e^{-2}(2)^3}{3!} = \mathbf{0.1804}$$

32. $\mu = 220/200 = 1.1$

$P(2 \text{ or more misprints}) = 1 - P(0 \text{ or } 1 \text{ misprint})$

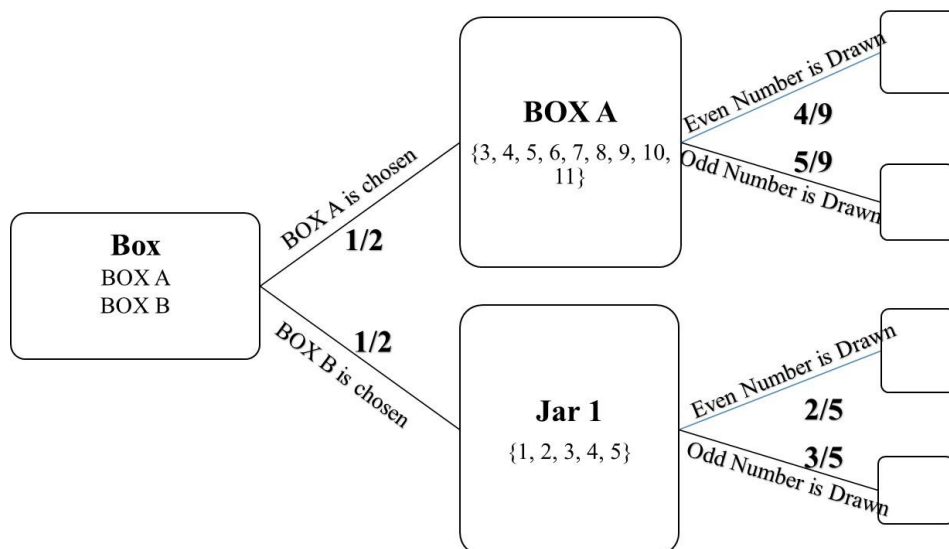
$$P(2 \text{ or more misprints}) = 1 - \sum_{x=0}^1 \frac{e^{-1.1}(1.1)^x}{x!} = \mathbf{0.301}$$

33.



Probability of getting white from the first jar $= \frac{3}{8} \left(\frac{5}{8} \right) + \frac{5}{8} \left(\frac{4}{8} \right) = \frac{35}{64}$

34.



Probability of getting even number $= \frac{1}{2} \left(\frac{4}{9} \right) + \frac{1}{2} \left(\frac{2}{5} \right) = \frac{19}{45}$

Probability that this number came from Box A $= \frac{\frac{1}{2} \left(\frac{4}{9} \right)}{\frac{19}{45}} = \frac{10}{19}$

35. Probability of picking a flunker:

$$P(F) = \frac{3\% \times 350 + 5\% \times 250 + 7\% \times 400}{350 + 250 + 400} = \frac{51}{1000}$$

$$P(M \cap F) = \frac{7\% \times 400}{350 + 240 + 400} = \frac{280}{990} = \frac{28}{99}$$

$$P(M|F) = \frac{P(M \cap F)}{P(F)} = \frac{\frac{28}{99}}{\frac{51}{1000}} = \mathbf{0.549}$$

36. $P(\text{man}) = P(M) = 10/30$; $P(\text{brown eyes}) = P(\text{BE}) = 1/2$

$$P(M \cap \text{BE}) = \frac{10}{30} \times \frac{1}{2} = \frac{1}{6}$$

$$P(M \cup \text{BE}) = P(M) + P(\text{BE}) - P(M \cap \text{BE})$$

$$P(M \cup \text{BE}) = \frac{10}{30} + \frac{1}{2} - \frac{1}{6} = \frac{2}{3}$$

37. Probability of hitting the target = $(1/4)(3/5) + (3/4)(2/5) + (1/4)(2/5)$
= **11/20**

Note: A hit, B miss + A miss, B hit + A hit, B hit = **11/20**

38. Probability of getting a reject = $1/1000$

Probability of getting a non-reject = $999/1000$

Probability of getting at least one reject = $1 - (999/1000)^{200} = \mathbf{0.18135}$

39. $P = (5C2)/(8C2) = \mathbf{5/14}$

40. $P = (3C2)/(8C2) = \mathbf{3/28}$

41. $P = (4C1 \times 6C1)/(10C2) = \mathbf{8/15}$

42. $P = (10C3)/(15C3) = \mathbf{24/91}$ (Note: Choose 3 out of $15 - 5 = 10$ non-defective)

43. $(13/52) \times (13/52) \times (13/52) \times (13/52) = \mathbf{1/256}$

44. $P = (4C4) \times (48C1)/(52C5) = \mathbf{1/54145}$

45.

$$P = \binom{8}{2; 2; 1; 1; 1; 1} = \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^1 = \frac{8!}{2! \times 2!} \left(\frac{1}{6}\right)^8$$

$$P = \frac{35}{5842} = \mathbf{0.006}$$

46.

$$P = \binom{5}{2; 1; 2} = (0.5)^2 (0.3)^1 (0.2)^2 = \frac{5!}{2! \times 1! \times 2!} (0.5)^2 (0.3)^1 (0.2)^2 = \mathbf{0.09}$$